ON HYPERFINITE $W^*$ ALGEBRAS

PAUL WILLIG

ABSTRACT. If $\mathcal{A}$ is a $W^*$ algebra on separable Hilbert space $H$, and if $\mathcal{A}(\lambda)$ are the factors in the direct integral decomposition of $\mathcal{A}$, then $f = \{\lambda | \mathcal{A}(\lambda) \text{ is hyperfinite}\}$ is $\mu$-measurable, and $\mathcal{A}$ is hyperfinite if and only if $\mathcal{A}(\lambda)$ is hyperfinite $\mu$-a.e.

Let $\mathcal{A}$ be a $W^*$ algebra on separable Hilbert space $H$. $\mathcal{A}$ is hyperfinite if there is an increasing sequence of finite dimensional $W^*$ subalgebras $\mathcal{A}_n$ of $\mathcal{A}$ whose union generates $\mathcal{A}$. Let $\mathcal{A} = \int_{\Lambda} \bigoplus \mathcal{A}(\lambda) \mu(d\lambda)$ denote the direct integral decomposition of $\mathcal{A}$ into factors, and let $f = \{\lambda | \mathcal{A}(\lambda) \text{ is hyperfinite}\}$. We prove in this paper that $f$ is $\mu$-measurable, and that $\mathcal{A}$ is hyperfinite if and only if $\mu(\Lambda - f) = 0$.

In dealing with direct integrals, we use the following notation (see [2], [3] for details). $K$ will denote the underlying separable Hilbert space of $H$. Letting $d$ denote a metric on $B(K)$ which induces the strong operator topology on bounded subsets of $B(K)$ [2, Lemma I.4.9], define $M(T) = d(T, 0)$. Then a bounded sequence $T_n \in B(K)$ converges strongly to 0 if and only if $M(T_n) \to 0$.

By $S$ we denote the unit ball of $B(K)$ taken with the strong $*$-topology, $S_n$, $n$ an integer, denotes the $n$-fold Cartesian product of $S$. Finally, let $B_n \in \mathcal{A}$ be a sequence in the unit ball of $\mathcal{A}$ such that $\{B_n(\lambda)\}$ is strong-$*$ dense in the unit ball of $\mathcal{A}(\lambda)$ $\mu$-a.e., and such that $B_n(\lambda)$ is strong-$*$ continuous in $\lambda$.

Before proving our main results, we consider the structure of a finite dimensional $W^*$ algebra $B$. Since any finite dimensional linear space of operators is strongly closed, $B$ is finite dimensional if and only if there is a finite set of operators $T_1, \ldots, T_n$ such that each product $T_i T_j$ and each adjoint $T_i^*$ is a linear combination of the $T_M$.

By the Kaplansky Density Theorem, it suffices for these operators to be strong limits of such linear combinations having bounded norms and coefficients in $C_0$, the set of complex numbers with rational real and imaginary parts. We apply this idea to define hyperfiniteness through countably many conditions. Indeed, if $\mathcal{A}$ is hyperfinite, for each $n$ there

Received by the editors October 23, 1972.

Key words and phrases. Hyperfinite, $W^*$ algebra, direct integral decomposition.
are \( n \) operators whose linear span is a finite dimensional \( W^\ast \)-algebra \( \mathcal{A}_n \) such that the \( \mathcal{A}_n \) form an increasing sequence (although not necessarily strictly increasing) whose union generates \( \mathcal{A} \). This explains the conditions in Theorem 1.

**Theorem 1.** Let \( \mathcal{J} = \{ \lambda | A(\lambda) \text{ is hyperfinite} \} \). Then \( \mathcal{J} \) is \( \mu \)-measurable.

**Proof.** Let \( \mathcal{P} = A \times \mathbb{R}_+, \mathcal{I}_n \), let \( \pi \) denote the projection of \( \mathcal{P} \) onto \( A \), and let \( T(n) = (T(n, 1), \cdots, T(n, n)) \) denote a typical element of \( \mathcal{I}_n \). Consider the following conditions on elements \([X, T(n)]\) of \( \mathcal{I}_n \):

1. \( T(n, m) \in \mathcal{A}(\lambda) \).
2. For some \( T = \sum_{k=1}^n a_k T(n, k), a_k \in C_0, T \in \mathcal{I}, \) and \( M(T(n, i) T(j, j) - T) < 1/r \).
3. For some \( T = \sum_{k=1}^n b_k T(n, k), b_k \in C_0, T \in \mathcal{I}, \) and \( M(T(n, i)^* - T) < 1/r \).
4. For some \( T = \sum_{k=1}^{n+1} c_k T(n+1, k), c_k \in C_0, T \in \mathcal{I}, \) and \( M(T(n, i) T) < 1/r \).
5. For some \( T = \sum_{k=1}^n d_k T(p, k), d_k \in C_0, T \in \mathcal{I}, \) and \( M(B_n(X) - T) < 1/r \).

It is easy to see that if \( \mathcal{J}' \) is the subset of \( \mathcal{P} \) for which condition (1) holds for every \( m \) and \( n \) and the remaining conditions hold for every \( r \), etc. for appropriate coefficients, then \( \mathcal{J}' \) is \( \mu \)-measurable and \( \pi(\mathcal{J}') \) differs from \( \mathcal{J} \) by a \( \mu \)-null set. Hence, by [2, Lemma 1.4.6], \( \mathcal{J} \) is \( \mu \)-measurable. Q.E.D.

**Theorem 2.** \( \mathcal{A} \) is hyperfinite if and only if \( \mu(\Lambda - \mathcal{J}) = 0 \).

**Proof.** Suppose \( \mathcal{A} \) is hyperfinite. Then for each \( n \) there is a finite dimensional \( W^\ast \)-subalgebra \( A_n \) of \( \mathcal{A} \) which is the linear span of \( T(n, 1), \cdots, T(n, n) \in \mathcal{A} \); these algebras form an increasing sequence whose union generates \( \mathcal{A} \). Now let \( A_n(\lambda) \) be the \( W^\ast \)-algebra generated by \( T(n, i)(\lambda), i=1, \cdots, n \). By [5, Lemma 1] it follows that \( A_n(\lambda) \) is an increasing sequence of finite dimensional \( W^\ast \)-algebras contained in \( A(\lambda) \) for \( \mu \)-a.e. \( \lambda \). Since \( B_n \in \mathcal{A} \), and the \( A_n \) generate \( \mathcal{A} \), a second application of [5, Lemma 1] shows that the \( A_n(\lambda) \) generate \( \mathcal{A}(\lambda) \) \( \mu \)-a.e. Thus \( \mu(\Lambda - \mathcal{J}) = 0 \).

Conversely, suppose that \( \mu(\Lambda - \mathcal{J}) = 0 \). Using the proof of Theorem 1 and [2, Lemma 1.4.7] we can construct an increasing sequence of finite dimensional \( W^\ast \)-subalgebras \( A_n \) of \( \mathcal{A} \) such that \( A_n(\lambda) \) generate \( \mathcal{A}(\lambda) \) \( \mu \)-a.e. It follows that \( \mathcal{A} \) is generated by the \( A_n \) and \( A^\ast \), the center of \( \mathcal{A} \). But \( A^\ast \) is hyperfinite [4, Lemma 2], and if \( C_n \) is an increasing sequence of finite dimensional \( W^\ast \)-algebras generating \( A^\ast \), then clearly for each \( n \) the algebra \( D_n \) generated by \( A_n \) and \( C_n \) is finite dimensional. Since \( \mathcal{A} \) is generated by the increasing sequence \( D_n \), the result is proved. Q.E.D.
We remark in conclusion that the idea of hyperfinite algebras was introduced by Murray and von Neumann in [1] to treat factors of type II$_1$. They proved that in this case the $\mathcal{A}_n$ could be chosen to be factors of type I$_{2n}$. This result has recently been extended to hyperfinite factors of types II$_{\infty}$ and III by E. J. Woods and G. Elliott (private communication). It is easy to see that our methods could then show that, modulo the center $\mathcal{Z}$, $\mathcal{A}$ hyperfinite is generated by a sequence of factors of type I$_{2n}$.

**Bibliography**


**Department of Mathematics, Stevens Institute of Technology, Hoboken, New Jersey 07030**