THE NUMBER OF CONTINUA

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Abstract. It is shown there are precisely $2^n$ topologically distinct continua of weight $n$ and power $m$ where $p \leq n \leq m$ and $p$ is the smallest cardinal for which there is a continuum of power $m$ and weight $p$. In particular, there are precisely $2^m$ topologically distinct continua of power $m$.

All spaces considered are assumed to be Hausdorff and all cardinals infinite. A continuum is a compact connected space. It follows from a classical result of P. Alexandroff and P. Urysohn ([1, p. 105]: The weight of a compact space never exceeds its power) that there are at most $2^m$ topologically distinct compact spaces of power $m$. (There are just $2^m$ collections, of cardinality at most $m$, of subsets of a set of cardinality $m$.) The following sharper result will be useful.

**Proposition 1.** There are at most $2^n$ topologically distinct compact spaces of weight $n$.

**Proof.** Suppose $\{X_\xi : \xi \in \Xi\}$ is a collection of topologically distinct compact spaces of weight $n$. For each $\xi \in \Xi$, let $\mathcal{B}_\xi$ be a base of cardinality $n$ for $X_\xi$. For each pair $U, V \in \mathcal{B}_\xi$ with $U \cap V = \emptyset$, let $f_{U, V}$ be an element of $C(X_\xi)$ such that $f_{U, V}|U = 1$ and $f_{U, V}|V = 0$. Let $\mathcal{F}_\xi$ denote the subset of $C(X_\xi)$ consisting of all these $f_{U, V}$'s together with all constant functions $f(x) = r$ with $r$ rational. Then $\text{card } \mathcal{F}_\xi \leq n$ and, by the Stone-Weierstrass Theorem, the smallest closed subring of $C(X_\xi)$ containing $\mathcal{F}_\xi$ is $C(X_\xi)$, where $C(X_\xi)$ is given the usual metric.

Now let $Z$ be a fixed discrete space of power $n$ and let $C^*(Z)$ have the usual metric. For each $\xi \in \Xi$, choose $f_\xi : Z \to X_\xi$ such that $f_\xi[Z]$ is dense in $X_\xi$, and let $F_\xi : C(X_\xi) \to C^*(Z)$ be the induced map of $f_\xi$; i.e., such that $F_\xi(f) = f \circ f_\xi$ for every $f \in C(X_\xi)$. Then each $F_\xi$ is a ring isomorphism so that, for $\xi \neq \xi'$, $F_\xi[C(X_\xi)]$ and $F_{\xi'}[C(X_{\xi'})]$ are nonisomorphic, and hence distinct, subrings of $C^*(Z)$. Furthermore, each $F_\xi$ is an isometry; in particular, since $C(X_\xi)$ is complete, $F_\xi[C(X_\xi)]$ is a closed subspace of
$C^*(Z)$. Thus the smallest closed subring of $C^*(Z)$ containing $F_{\xi}[[F]]$ is $F_{\xi}[C(X_{\xi})]$ for every $\xi \in \Xi$. Thus for $\xi \neq \xi'$, $F_{\xi}[[F]] \neq F_{\xi'}[[F]]$. Consequently, \{ $F_{\xi}[[F]] : \xi \in \Xi$ \} is a collection of distinct subsets of $C^*(Z)$ of cardinality at most $n$. But $C^*(Z)$ has cardinality $2^n$ and hence at most $2^n$ subsets of cardinality at most $n$. Thus card $\Xi \leq 2^n$.

**Proposition 2.** For every cardinal $m \geq 2^{\aleph_0}$, there are $2^m$ topologically distinct continua of power $m$ and weight $m$.

**Proof.** Let $L$ denote the long line constructed on the set $S$ of all ordinals $\beta \leq \omega(m)$, the initial ordinal of cardinality $m$. We regard $S$ as a subset of $L$. For each $\beta \in S$, let $(X_{\beta}, x_{\beta})$ be either $(I^2, (0,0))$ or $(I^3, (0,0,0))$, where $I = [0,1] \subseteq R$. Let $X$ be the space obtained by attaching each $X_{\beta}$ to $L$ by identifying $x_{\beta} \in X_{\beta}$ with $\beta \in S$, and weakening the usual quotient topology by requiring that any neighborhood of a limit ordinal $\gamma \in S$ contains $\bigcup \{X_{\beta} : \alpha < \beta < \gamma \}$ for some $\alpha < \gamma$. Then $X$ is a continuum of power $m$ and weight $m$.

Now suppose $X'$ were another such space constructed in the same way but with a conceivably different choice of the $(X'_{\beta}, x'_{\beta})$'s, and suppose $f : X \rightarrow X'$ were an onto homeomorphism. Let $Y$ denote the set of all points of $X$ at which the dimension is 1; then $L = \overline{Y}$ and $S = \overline{Y} \cap (X - Y)$. Similarly, $L' = \overline{Y}'$ and $S' = \overline{Y}' \cap (X' - Y')$ in $X'$. It is immediate that $f[I] = Y'$, and hence $f[L] = L'$ and $f[S] = S'$. But then $f|L$ is monotone, so that $f|S$ is order-preserving and hence the identity. Now consider the subspace $Z = \bigcup \{X_{\beta} : \beta \in S\}$ of $X$, which is the disjoint union of the connected subspaces $X_{\beta}$, and the analogous subspace $Z'$ of $X'$. Because $Z = (X - L) \cup S$ and $Z' = (X' - L') \cup S'$, it follows that $f[Z] = Z'$; in particular, $f$ maps each $X_{\beta}$ onto some $X'_{\beta}$. Because $f|S$ is the identity, it follows that $f[X_{\beta}] = X'_{\beta}$ for every $\beta \in S$. Therefore, since $I^2$ and $I^3$ are not homeomorphic, it follows that $X'_{\beta} = X_{\beta}$ for every $\beta \in S$.

Finally, since there are $2^m$ different ways to choose the $X_{\beta}$'s there are $2^m$ topologically distinct continua of power $m$ and weight $m$.

**Proposition 3.** For every cardinal $m \geq 2^{\aleph_0}$, let $p$ be the smallest cardinal for which there is a continuum of power $m$ and weight $p$. Then for every cardinal $n$ with $p \leq n \leq m$, there are $2^n$ topologically distinct continua of power $m$ and weight $n$.

**Proof.** Let $K$ be a continuum of power $m$ and weight $p$. For any cardinal $n$ with $p \leq n \leq m$, let $L$ be the long line constructed on the set of all ordinals $\beta \leq \omega(n)$. Construct $X$ as in Proposition 2; but take $X_0$ to be $K \times I^2$ and take $X_{\beta}$, for $\beta > 0$, to be either $I^2$ or $I^3$. The argument is then similar to Proposition 2.
Remarks. For the case $m = 2^\aleph_0$, the requirement that $X_0 = K \times I^2$ can be dispensed with. Furthermore, for the case $m = 2^\aleph_0$, $n = \aleph_0$, all of the constructed continua can be embedded in the plane by choosing $X_q$ to be either $I^2$ or an annulus. If $m = 2^q$ for some $q$, then $p$ is simply the smallest such $q$; for then $I^p$ is a continuum of power $m$ and weight $p$. In particular, if we assume the Generalized Continuum Hypothesis, then the only continua of power $m$ other than those of weight $m$ are (for nonlimit cardinals $m$) the $2^n$ topologically distinct continua of weight $n$ where $2^n = m$.

The authors wish to express their gratitude to S. Mrowka and the referee for their helpful suggestions.

References