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EXCEPTIONAL VALUES OF ANALYTIC FUNCTIONS

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ABSTRACT. Let $g(z)$ be an analytic function with an isolated singularity at z_0 and let $f(z)$ be a nonanalytic multianalytic function with a nonessential isolated singularity at z_0. It is shown that if $g(z)-f(z)$ does not vanish in some deleted neighborhood of z_0, then z_0 is a nonessential singularity of $g(z)$.

Suppose that $g(z)$ is analytic with an isolated singularity at z_0, where z_0 is an arbitrary complex number, finite or infinite. Let $\bar{z}$ denote the complex conjugate of z. We offer the following result. If $g(z)-\bar{z}$ or $g(z)-1/\bar{z}$ does not vanish in some deleted neighborhood of z_0, then z_0 is not an essential singularity of $g(z)$. This result follows from a characterization of bianalytic functions which do not vanish in some deleted neighborhood of an isolated singularity [**2**]. In case $g(z)$ is entire and $z_0=\infty$, the above result also follows from a theorem of M. B. Balk [**1**] which characterizes polyentire functions which do not vanish in some deleted neighborhood of $z_0=\infty$.

The foregoing result leads us to make the following definition. Let z_0 be an arbitrary complex number, finite or infinite. We define $E(z_0)$ to be the collection of all complex valued functions $f(z)$ defined in some deleted neighborhood of z_0 with the following property: If $g(z)$ is any analytic function with an isolated singularity at z_0 such that $g(z)-f(z)$ does not vanish in some deleted neighborhood of z_0, then z_0 is not an essential singularity of $g(z)$.

We are thus led to the problem of determining all the functions which belong to $E(z_0)$. We introduce a class of functions called multianalytic functions. It is shown that if $f(z)$ is a nonanalytic multianalytic function with a nonessential isolated singularity at z_0, then $f(z)\in E(z_0)$. If $g(z)$ is analytic with an isolated singularity at z_0, then as one application we obtain necessary and sufficient conditions in order that $(g(z))^- \in E(z_0)$, where $(g(z))^-$ denotes the complex conjugate of $g(z)$. As a further application we deduce the existence of fixed points for a certain class of analytic functions.

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We now introduce some notation and give some definitions. Let z_0 be an arbitrary complex number, finite or infinite, and let $0<R<+\infty$. If z_0 is finite, then $A(z_0, R)$ will denote the set of all finite complex numbers z such that $0<|z-z_0|<R$. If $z_0=\infty$, then $A(z_0, R)$ will denote the set of all finite complex numbers z such that $R<|z|<+\infty$. We also let $N(z_0, R)= A(z_0, R)\cup\{z_0\}$.

A function $f(z)$ defined on $A(z_0, R)$ is termed multianalytic on $A(z_0, R)$ if and only if there is some nonnegative integer n and a sequence $f_k(z)$ of functions analytic on $A(z_0, R)$ for $k\geqq -n$ such that

$$(1) \qquad f(z) = \sum_{k=-n}^{+\infty} ((z - z_0)^{-})^k f_k(z),$$

for all z in $A(z_0, R)$, where $(z-z_0)^{-}$ denotes the complex conjugate of $(z-z_0)$. If $z_0=\infty$ then in the series in equation (1) it is understood that the term $((z-z_0)^{-})^k$ is replaced by $1/\bar{z}^k$ for each $k\geqq -n$. It is also understood that the series in equation (1) is almost uniformly convergent on $A(z_0, R)$, that is the series is uniformly convergent on every nonempty closed subset of $A(z_0, R)$. If z_0 is any complex number, finite or infinite, then a function $f(z)$ is said to be multianalytic at z_0 if and only if there is some $0<R<+\infty$ so that $f(z)$ is multianalytic on $A(z_0, R)$.

Next let $f(z)$ be multianalytic on $A(z_0, R)$ and represented on $A(z_0, R)$ by equation (1). We shall show later that the functions $f_k(z)$ in equation (1) are uniquely determined on $A(z_0, R)$ by $f(z)$. For each $k\geqq -n$, we define $d_k=d(f_k)$, the order of $f_k(z)$ at z_0, as follows. If $f_k(z)\equiv 0$ on $A(z_0, R)$, then $d_k=-\infty$. If z_0 is an essential singularity of $f_k(z)$, then $d_k=+\infty$. Now suppose that $f_k(z)\not\equiv 0$ on $A(z_0, R)$ and that z_0 is a nonessential singularity of $f_k(z)$. If $z_0\neq\infty$, then d_k is that unique integer such that $(z-z_0)^{d_k}f_k(z)$ is analytic and not zero at z_0. If $z_0=\infty$, then d_k is that unique integer such that $f_k(z)/z^{d_k}$ is analytic and not zero at $z_0=\infty$. We then define the order d of $f(z)$ at z_0 to be the least upper bound of the numbers d_k-k for $k\geqq -n$. Note that d is an integer or $\pm\infty$. Observe that $d=-\infty$ if and only if $f(z)\equiv 0$ on $A(z_0, R)$. The point z_0 is termed an essential or a nonessential singularity of $f(z)$ according as $d=+\infty$ or $d\neq+\infty$.

Note that if $f(z)$ is multianalytic at z_0 of order d and if $h(z)$ is a Möbius transformation, then $f[h(z)]$ is multianalytic at $h^{-1}(z_0)$ of the same order d.

We now need some preliminary lemmas.

LEMMA 1. *Let $f(z)$ be multianalytic on $A(\infty, R)$ and represented on $A(\infty, R)$ by equation* (1) *where $z_0=\infty$. Then for $\rho>R$, the series*

$$(2) \qquad f(z, \rho) = \sum_{k=-n}^{+\infty} \frac{z^k f_k(z)}{\rho^{2k}},$$

is almost uniformly convergent on $R<|z|\leqq\rho$.

PROOF. Observe that the general term of the series in equation (2) can be written in the form $z^k f_k(z)/\rho^{2k} = (z\bar{z}/\rho^2)^k f_k(z)/\bar{z}^k$. For $k \geqq 0$, the sequence $(z\bar{z}/\rho^2)^k$ is decreasing and uniformly bounded for $R < |z| \leqq \rho$. Hence by Abel's theorem the series in equation (2) is almost uniformly convergent in $R < |z| \leqq \rho$ as claimed.

Note that the auxiliary function $f(z, \rho)$ defined by equation (2) is continuous on $R < |z| \leqq \rho$ and analytic on $R < |z| < \rho$ and that $f(z, \rho) = f(z)$ for $|z| = \rho > R$.

We now have the following uniqueness result to which we appealed earlier.

LEMMA 2. *Let $f(z)$ be multianalytic on $A(z_0, R)$ and represented on $A(z_0, R)$ by equation* (1). *Then the functions $f_k(z)$ in equation* (1) *are uniquely determined on $A(z_0, R)$ by $f(z)$.*

PROOF. It suffices to show that if $f(z) \equiv 0$ on $A(z_0, R)$, then $f_k(z) \equiv 0$ on $A(z_0, R)$ for each $k \geqq -n$. We need only consider the case when $z_0 = \infty$. Let $f(z)$ be represented on $A(\infty, R)$ by equation (1) where $z_0 = \infty$. From equation (2) we see that $f(z, \rho) = 0$ for $|z| = \rho > R$. Hence $f(z, \rho) = 0$ for $R < |z| \leqq \rho$. Let $z \in A(\infty, R)$ be fixed. From Lemma 1, it follows that the Laurent series $g(w) = \sum z^k f_k(z)/w^{2k}$, $k \geqq -n$, is convergent in case w is real and $w > |z|$. Hence the Laurent series $g(w)$ is almost uniformly convergent on $A(\infty, |z|)$. Also $g(w) = 0$ in case w is real and $w > |z|$. Hence $g(w) \equiv 0$ on $A(\infty, |z|)$ by the principle of permanence. From the uniqueness of the Laurent series expansion for $g(w)$ it follows that $z^k f_k(z) = 0$ for $k \geqq -n$. Since z is an arbitrary point of $A(\infty, R)$, it follows that $z^k f_k(z) \equiv 0$ on $A(\infty, R)$ for each $k \geqq -n$ and the result follows.

We now need two technical lemmas regarding the auxiliary function $f(z, \rho)$.

LEMMA 3. *Let $f(z)$ be multianalytic on $A(\infty, R)$ and represented on $A(\infty, R)$ by equation* (1) *where $z_0 = \infty$. Assume that $f_0(z) \equiv 0$ on $A(\infty, R)$ and that $f(z)$ is of order $d \neq \pm\infty$ at $z_0 = \infty$. Then there exists a function $P(z)$ analytic and not identically constant on $A(0, 1)$ such that $f(\rho z, \rho)/\rho^d z^d \to P(z)$ as $\rho \to +\infty$ almost uniformly in $A(0, 1)$.*

PROOF. If $d_k \neq -\infty$, then $f_k(z)/z^{d_k}$ is analytic and not zero at $z_0 = \infty$. In this case let $a_k \neq 0$ be the value of $f_k(z)/z^{d_k}$ at $z_0 = \infty$. Clearly $d_k \neq -\infty$ for at least one $k \geqq -n$. Now

$$\frac{f(\rho z, \rho)}{\rho^d z^d} = \sum z^{2k} (\rho z)^{d_k - k - d} \frac{f_k(\rho z)}{(\rho z)^{d_k}}, \qquad k \geqq -n, d_k \neq -\infty,$$

for $R/\rho < |z| < 1$. From the above equation it seems reasonable to suppose that $f(\rho z, \rho)\rho^d z^d \to \sum a_k z^{2k}$, $k \geqq -n$, $d_k - k = d$, almost uniformly with respect to z in $A(0, 1)$ as $\rho \to +\infty$. We shall show that the above result is true. Let $\sigma > R$ be fixed. Since the series in equation (1) is uniformly convergent on $|z| = \sigma$, then there is some positive constant M so that $|f_k(z)/\bar{z}^k| \leqq M$ for $|z| = \sigma$ and $k \geqq -n$. Hence $|f_k(z)/z^{d_k}| \leqq M/\sigma^{d_k - k}$ for $|z| = \sigma$ when $d_k \neq -\infty$. Since $f_k(z)/z^{d_k}$ is analytic on $A(\infty, R)$ then by the maximum modulus principle we see that $|f_k(z)/z^{d_k}| \leqq M/\sigma^{d_k - k}$ for $\sigma \leqq |z| \leqq +\infty$ whenever $d_k \neq -\infty$. Hence $|a_k| \leqq M/\sigma^{d_k - k}$ when $d_k \neq -\infty$. Thus $|a_k| \leqq M/\sigma^d$ when $d_k - k = d$. Thus the function $P(z) = \sum a_k z^{2k}$, $k \geqq -n$, $d_k - k = d$, is analytic and not identically constant in $A(0, 1)$. We can now write $f(\rho z, \rho)/\rho^d z^d - P(z) = \sum_1 z^{2k}(\rho z)^{d_k - k - d} f_k(\rho z)^{d_k}/(\rho z)^{d_k} + \sum_2 z^{2k}(f_k(\rho z)/(\rho z)^{d_k} - a_k)$, for $R/\rho < |z| < 1$, where in the first series $\sum_1$ we sum over all $k \geqq -n$ such that $d_k \neq -\infty$ and $d_k - k \leqq d - 1$ and in the second series $\sum_2$ we sum over all $k \geqq -n$ such that $d_k - k = d$. We now proceed to obtain upper bounds for the two series $\sum_1$ and $\sum_2$. First note that $|f_k(\rho z)^{d_k}| \leqq M/\sigma^{d_k - k}$ for $\sigma \leqq \rho|z| \leqq +\infty$ when $d_k \neq -\infty$. Thus if $\sigma/\rho < |z| < 1$ we see that

$$\left|\sum\nolimits_1\right| \leqq \sum\nolimits_1 |z|^{2k} |\rho z|^{d_k - k - d} M/\sigma^{d_k - k} = M\sigma^{-d} \sum\nolimits_1 |z|^{2k} |\sigma/\rho z|^{d - (d_k - k)}$$
$$\leqq M\sigma^{-d} \sum\nolimits_1 |z|^{2k} |\sigma/\rho z| \leqq M\,|z|^{-2n-1}/\rho\sigma^{d-1}(1 - |z|^2).$$

Next note that $|f_k(z)/z^{d_k} - a_k| \leqq 2M/\sigma^{d_k - k}$ for $\sigma \leqq |z| \leqq +\infty$ when $d_k \neq -\infty$. Now the function $f_k(z)/z^{d_k} - a_k$ is analytic on $A(\infty, R)$ and vanishes at $z_0 = \infty$. Hence by Schwarz's lemma we deduce that $|f_k(z)/z^{d_k} - a_k| \leqq 2M/|z|\sigma^{d_k - k - 1}$ for $\sigma \leqq |z| \leqq +\infty$ when $d_k \neq -\infty$. Hence

$$|f_k(\rho z)^{d_k}/(\rho z)^{d_k} - a_k| \leqq 2M/\rho\,|z|\,\sigma^{d-1} \quad \text{for } \sigma \leqq \rho\,|z| \leqq +\infty$$

when $d_k - k = d$. Thus if $\sigma/\rho < |z| < 1$ we see that

$$\left|\sum\nolimits_2\right| \leqq \sum\nolimits_2 |z|^{2k}\, 2M/\rho\,|z|\,\sigma^{d-1} \leqq 2M\,|z|^{-2n-1}/\rho\sigma^{d-1}(1 - |z|^2).$$

From the foregoing upper bounds for $\sum_1$ and $\sum_2$ we deduce that

$$|f(\rho z, \rho)/\rho^d z^d - P(z)| \leqq 3M\,|z|^{-2n-1}/\rho\sigma^{d-1}(1 - |z|^2)$$

for $\sigma/\rho < |z| < 1$. The result now follows.

LEMMA 4. *Let $f(z)$ be multianalytic on $A(\infty, R)$ and of order $d \neq \pm\infty$ at $z_0 = \infty$. Then there is an integer m and a function $h(z)$ analytic and not identically zero on $A(\infty, R)$ with a nonessential singularity at $z_0 = \infty$ such that $\rho^m f(z, \rho) \to h(z)$ as $\rho \to +\infty$ almost uniformly on $A(\infty, R)$.*

PROOF. Let $f(z)$ be represented on $A(\infty, R)$ by equation (1) where $z_0 = \infty$. There is some integer $s \geqq -n$ so that $f_k(z) \equiv 0$ on $A(\infty, R)$ for $-n \leqq$

$k\leqq s-1$ but $f_s(z)\not\equiv 0$ on $A(\infty, R)$. Hence $\rho^{2s}f(z, \rho)=\sum z^k f_k(z)/\rho^{2(k-s)}$, $k\geqq s$, for $R<|z|\leqq\rho$. From the above equation it seems reasonable to suppose that $\rho^{2s}f(z, \rho)\to z^s f_s(z)$ as $\rho\to+\infty$ almost uniformly on $A(\infty, R)$. We shall show that this is the case. Let $\sigma>R$ be fixed. Since the series in equation (1) is uniformly convergent on $|z|=\sigma$, there is some $M>0$ so that $|f_k(z)/\bar{z}^k|\leqq M$ for $|z|=\sigma$ and $k\geqq s$. Hence $|f_k(z)/z^{d_k}|\leqq M/\sigma^{d_k-k}$ for $|z|=\sigma$ and $k\geqq s$ when $d_k\neq-\infty$. Now $f_k(z)/z^{d_k}$ is analytic on $A(\infty, R)$ so that by the maximum modulus principle we deduce that $|f_k(z)/z^{d_k}|\leqq M/\sigma^{d_k-k}$ for $\sigma\leqq|z|\leqq+\infty$ and $k\geqq s$ when $d_k\neq-\infty$. Thus if $k\geqq s$ and $d_k\neq-\infty$ and $\sigma\leqq|z|<+\infty$ we see that

$$\begin{aligned}|z^k f_k(z)| &= |z^{d_k+k}f_k(z)/z^{d_k}| \leqq |z|^{d_k+k}\, M/\sigma^{d_k-k}\\ &= M\,|z|^{2k}\,|z/\sigma|^{d_k-k} \leqq M\,|z|^{2k}\,|z/\sigma|^{d} = M\,|z|^{d+2k}/\sigma^{d}.\end{aligned}$$

Thus for all $k\geqq s$ we see that $|z^k f_k(z)|\leqq M|z|^{d+2k}/\sigma^d$ when $\sigma\leqq|z|<+\infty$. From the above estimate it is now easy to see that $|\rho^{2s}f(z, \rho)-z^s f_s(z)|\leqq M|z|^{d+2s+2}/\sigma^d(\rho^2-|z|^2)$ for $R<\sigma\leqq|z|<\rho$. The result now follows.

If $f(z)$ is an arbitrary complex valued function which is continuous and never zero on $|z|=\rho>0$, we define $\Delta_\rho f(z)$ to be $1/2\pi$ times the change in the argument of $f(z)$ around the positively oriented circumference $|z|=\rho$.

We are now in a position to establish our main result.

THEOREM. *Let $f(z)$ be multianalytic and not analytic on $A(z_0, R)$ with a nonessential singularity at z_0. Then $f(z)\in E(z_0)$.*

PROOF. We need only consider the case when $z_0=\infty$. Let $f(z)$ be multianalytic and not analytic on $A(\infty, R)$ and of order $d\neq\pm\infty$ at $z_0=\infty$ and let $g(z)$ be analytic on $A(\infty, R)$ such that $g(z)-f(z)$ does not vanish on $A(\infty, R)$. We wish to show that $z_0=\infty$ is not an essential singularity of $g(z)$. We shall suppose that $z_0=\infty$ is an essential singularity of $g(z)$ and arrive at a contradiction. We need only consider the case when $d=0$. Further if $f(z)$ is represented on $A(\infty, R)$ by equation (1) where $z_0=\infty$, we need only consider the case when $f_0(z)\equiv 0$ on $A(\infty, R)$. There is some integer p so that $\Delta_\rho[g(z)-f(z)]=p$ for all $\rho>R$. Hence $\Delta_\rho[g(z)-f(z, \rho)]=p$ for all $\rho>R$. From Lemma 4, there is an integer m and a function $h(z)$ analytic and not identically zero on $A(\infty, R)$ with a nonessential singularity at $z_0=\infty$ such that $\rho^m f(z, \rho)\to h(z)$ as $\rho\to+\infty$ almost uniformly on $A(\infty, R)$. Now choose some fixed $\sigma>R$ so that $g(z)\neq 0$ and $h(z)\neq 0$ and $g(z)\neq h(z)$ for all z on $|z|=\sigma$. Now $\rho^m g(z)-\rho^m f(z, \rho)\to\infty$ or $g(z)-h(z)$ or $-h(z)$ as $\rho\to+\infty$ uniformly on $|z|=\sigma$ according as $m\geqq 1$ or $m=0$ or $m\leqq-1$. It is easy to see that there is some integer q and some $\lambda>\sigma$ so that $\Delta_\sigma[g(z)-f(z, \rho)]=q$ for all $\rho>\lambda$. We thus see that the equation $g(z)=f(z, \rho)$ has exactly $p-q=r$ solutions, counting multiplicities, in $\sigma<|z|<\rho$

for all $\rho>\lambda$. It follows that each of the equations $g(\rho z)=f(\rho z, \rho)$ and $g(\rho z)=f(\rho z, 2\rho)$ has at most r solutions in $\sigma/\lambda<|z|<1$ for all $\rho>\lambda$. From Lemma 3, there is a function $P(z)$ analytic and not constant in $A(0, 1)$ such that $f(\rho z, \rho)\to P(z)$ as $\rho\to+\infty$ almost uniformly on $A(0, 1)$. Now $P(z/2)-P(z)\not\equiv 0$ on $A(0, 1)$. Hence there is some $\sigma/\lambda<a<b<1$ so that $P(z)\neq 0$ and $P(z/2)-P(z)\neq 0$ for $a\leqq|z|\leqq b$. Now $f(\rho z, 2\rho)-f(\rho z, \rho)\to P(z/2)-P(z)$ as $\rho\to+\infty$ almost uniformly on $A(0, 1)$. Hence there is some $\mu>\lambda$ so that $f(\rho z, 2\rho)-f(\rho z, \rho)\neq 0$ and $f(\rho z, \rho)\neq 0$ for $a\leqq|z|\leqq b$ and $\rho>\mu$. Hence we can find positive constants A, B, and C so that $A>B|f(\rho z, 2\rho)-f(\rho z, \rho)|>2|f(\rho z, \rho)|>2C$ for $a\leqq|z|\leqq b$ and $\rho>\mu$. We shall utilize these inequalities later in the proof. Thus for $\rho>\mu$, the function

$$[g(\rho z) - f(\rho z, \rho)]/[f(\rho z, 2\rho) - f(\rho z, \rho)]$$

is analytic on $a<|z|<b$ and assumes each of the values zero and one at most r times in $a<|z|<b$. Let H denote the collection of these functions for $\rho>\mu$. This family H is a family of functions which is analytic on $a<|z|<b$ and quasinormal on $a<|z|<b$ of order not exceeding r **[3]**. Hence there is some $a<c<b$ and some sequence $\rho_n>\mu$ diverging to $+\infty$ so that the sequence

$$h_n(z) = [g(\rho z) - f(\rho z, \rho)]/[f(\rho z, 2\rho) - f(\rho z, \rho)] \quad \text{for } \rho = \rho_n$$

is either uniformly convergent on $|z|=c$ or is uniformly divergent to ∞ on $|z|=c$. In either case we shall arrive at a contradiction by showing that $z_0=\infty$ is not an essential singularity of $g(z)$. First suppose that the sequence $h_n(z)$ is uniformly convergent on $|z|=c$. Hence there is some $K>0$ so that $|h_n(z)|\leqq K$ for $|z|=c$. Hence $|g(z)|<K(A/2+A/B)$ for $|z|=c\rho_n$. Hence $z_0=\infty$ is not an essential singularity of $g(z)$. Next suppose that the sequence $h_n(z)$ is uniformly divergent to ∞ on $|z|=c$. Hence there is some integer n_0 so that $|h_n(z)|>B$ for $|z|=c$ and $n\geqq n_0$. Thus

$$|g(\rho z) - f(\rho z, \rho)| > B\,|f(\rho z, 2\rho) - f(\rho z, \rho)| > 2\,|f(\rho z, \rho)| > 2C$$

for $|z|=c$ and $\rho=\rho_n$ and $n\geqq n_0$. From the above inequalities we make two deductions. First it follows that $|g(z)|>C$ for $|z|=c\rho_n$ and $n\geqq n_0$. Second it follows that $\Delta_c g(\rho z)=\Delta_c[g(\rho z)-f(\rho z, \rho)]\leqq p$ for $\rho=\rho_n$ and $n\geqq n_0$ so that $g(z)$ does not vanish in some deleted neighborhood of $z_0=\infty$. Hence $g(z)$ does not have an essential singularity at $z_0=\infty$. This completes the proof of the theorem.

As immediate consequences of the above theorem we offer the following corollaries.

COROLLARY 1. *Let $g(z)$ be nonconstant and analytic on $A(z_0, R)$. Then $(g(z))- \in E(z_0)$ if and only if z_0 is a nonessential singularity of $g(z)$.*

PROOF. If z_0 is a nonessential singularity of $g(z)$, then from the above theorem we have that $(g(z))^- \in E(z_0)$. Next suppose that z_0 is an essential singularity of $g(z)$. Let $f(z)=g(z)+1$ for $z \in A(z_0, R)$. Then $f(z)$ is analytic on $A(z_0, R)$ with an essential singularity at z_0. Clearly $f(z)-(g(z))^-$ does not vanish on $A(z_0, R)$. Hence $(g(z))^- \notin E(z_0)$. This proves the corollary.

COROLLARY 2. *Let $g_1(z), \cdots, g_n(z)$ and $h_1(z), \cdots, h_n(z)$ be analytic on $A(z_0, R)$ with a nonessential singularity at z_0. If $f(z)=g_1(z)(h_1(z))^- + \cdots + g_n(z)(h_n(z))^-$ is not analytic on $A(z_0, R)$, then $f(z) \in E(z_0)$.*

COROLLARY 3. *Let $z_0 \neq \infty$ and $R>0$ and let a_{rs} be a sequence of complex numbers for r, $s \geqq 0$. If $f(z)=\Sigma\, a_{rs}(z-z_0)^r((z-z_0)^-)^s$ is defined and not analytic in $|z-z_0|<R$, then $f(z) \in E(z_0)$.*

In [4], P. C. Rosenbloom showed that every entire transcendental function possesses fixed points of exact order one or two. We obtain an extension of the above result to a certain class of analytic functions.

COROLLARY 4. *Let $g(z)$ be analytic on $A(\infty, R)$ with an essential singularity at $z_0=\infty$ and suppose that $g(z)$ is real valued on the real axis. Then $g(z)$ possesses a fixed point of exact order one or two in $A(\infty, R)$.*

PROOF. First $(g(z))^- = g(\bar{z})$ for all z in $A(\infty, R)$. From Corollary 1, there is some ζ in $A(\infty, R)$ so that $g(\zeta)=\bar{\zeta}$. Hence $g(\bar{\zeta})=\zeta$. Hence $g[g(\bar{\zeta})]$ is defined and $g[g(\bar{\zeta})]=\bar{\zeta}$ so that $\bar{\zeta}$ is a fixed point of $g(z)$ of exact order one or two in $A(\infty, R)$.

REFERENCES

1. M. B. Balk, *Entire polyanalytic functions with a bounded set of zeros*, Izv. Akad. Nauk Armjan. SSR Ser. Mat. **1** (1966), no. 5, 341–357. MR **34** #6127.

2. P. Krajkiewicz, *Bianalytic functions with exceptional values*, Proc. Amer. Math. Soc. **38** (1973), 75–79.

3. P. Montel, *Leçons sur les familles normales de fonctions analytiques et leurs applications*, Gauthier-Villars, Paris, 1927.

4. P. C. Rosenbloom, *The fix-points of entire functions*, Medd. Lunds Univ. Mat. Sem. Tome Suppl. (1952), 186–192. MR **14**, 546.

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