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ISOMETRIC EMBEDDING OF A COMPACT RIEMANNIAN MANIFOLD INTO EUCLIDEAN SPACE[1]

HOWARD JACOBOWITZ

ABSTRACT. An isometric immersion of an n-dimensional compact Riemannian manifold with sectional curvature always less than λ^{-2} into Euclidean space of dimension $2n-1$ can never be contained in a ball of radius λ. This generalizes and includes results of Tompkins and Chern and Kuiper.

Tompkins [4] proved that the flat n-dimensional torus could not be isometrically embedded into E, the Euclidean space of dimension $2n-1$. Chern and Kuiper [1] and Otsuki [3] generalized this and showed that a compact n-dimensional manifold whose sectional curvatures are everywhere nonpositive also cannot be isometrically embedded in E. Apparently these results also hold for immersions. The purpose of this note is to point out that a standard proof of this nonembeddability result essentially establishes a more general quantitative version.

THEOREM. *Let E be Euclidean space of dimension $2n-1$ and M a compact n-dimensional Riemannian manifold whose sectional curvatures are everywhere less than some constant λ^{-2}. Then no isometric immersion of M into E is contained in a ball of radius λ.*

To prove the theorem, we adapt the proof for nonpositive curvature given in Kobayashi and Nomizu [2, pp. 26–29]. Note the theorem includes the results mentioned above.

Let $f: M \to E$ be an isometric embedding. Identify the tangent space T_xM for $x \in M$ with its realization as a linear subspace of E. Denote the length and inner product of vectors in E by $|X|$ and $\langle X, Y\rangle$. To prove the theorem, let us assume such an isometric immersion did exist. We can assume $|f(x)| \leqq \lambda$ while $|f(x_0)| = \lambda$ for some x_0 and all x in M. Thus $\langle f(x_0), X\rangle = 0$ for all X in $T_{x_0}M$. Let $L(X, Y)$ denote the second fundamental form at x_0 of M in E. For X and Y in $T_{x_0}M$, $L(X, Y)$ is a vector in E orthogonal to M at x_0. The sectional curvature of the two-plane spanned by

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linearly independent vectors X and Y in $T_{x_0}M$ is given by

$$K(X, Y) = (|X|^2 |Y|^2 - \langle X, Y\rangle^2)^{-1} \cdot (\langle L(X, X), L(Y, Y)\rangle - \langle L(X, Y), L(X, Y)\rangle).$$

For a proof of the theorem it suffices to find two vectors X and Y such that $K(X, Y) \geqq \lambda^{-2}$.

Differentiating $\langle f(x), f(x)\rangle$ twice at the maximum $x = x_0$, one obtains $\langle L(X, X), \nu\rangle \leqq -\lambda^{-1}|X|^2$ for the unit normal $\nu = \lambda^{-1} f(x_0)$ and all $X \in T_{x_0}M$. In particular $L(X, X) = 0$ only for $X = 0$. Now for any symmetric bilinear form $B: \boldsymbol{R}^n \times \boldsymbol{R}^n \to \boldsymbol{R}^{n-1}$ with $B(X, X) = 0$ only for $X = 0$, one can find linearly independent vectors X and Y such that $B(X, X) = B(Y, Y)$ and $B(X, Y) = 0$. Indeed just solve $B(Z, Z) = 0$ where Z is allowed to be complex. This observation is due to T. A. Springer; see [**2**]. Let X and Y be two such vectors for L. Since $\langle L(X, X), \nu\rangle \leqq -\lambda^{-1}|X|^2$ while $L(X, X) = L(Y, Y)$ one has $\langle L(X, X), L(Y, Y)\rangle \geqq \lambda^{-2}|X|^2|Y|^2$. This together with $L(X, Y) = 0$ implies the inequality $K(X, Y) \geqq \lambda^{-2}$ and so proves the theorem.

If for the manifold M we only knew that for each x, T_xM contained a q-dimensional subspace T'_x with the property that the sectional curvature of each two plane in T'_x was greater than λ^{-2}, then we could conclude that no isometric immersion of M into E^{n+q-1} is contained in a ball of radius λ. This result, for nonpositive curvatures, is due to Chern and Kuiper [**1**].

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Department of Mathematics, Rice Uiversity, Houston, Texas 77001