COMMUTATIVE EXTENSIONS BY CANONICAL MODULES ARE GORENSTEIN RINGS

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Abstract. Reiten has demonstrated that the trivial Hochschild extension of a Cohen-Macaulay local ring by a canonical module is a Gorenstein local ring. Here it is proved that any commutative extension of a Cohen-Macaulay local ring by a canonical module is a Gorenstein ring. Also Gorenstein extensions of a local Cohen-Macaulay ring by a module are studied.

Introduction. Suppose $A$ is a commutative ring and that $M$ is an $A$-module. A commutative extension of $A$ by $M$ is an exact sequence of abelian groups

$$0 \rightarrow M \xrightarrow{i} E \xrightarrow{\pi} A \rightarrow 0,$$

where $E$ is a commutative ring, the map $\pi$ is a ring homomorphism and the $A$-module structure on $M$ is related to $(E, i, \pi)$ by the equations

$$ei(x) = i(\pi(e)x), \text{ for all } e \in E \text{ and all } x \in M.$$

The $i$ identifies $M$ with an ideal of square zero in $E$. (On the other hand if $\mathfrak{J}$ is an ideal of square zero in $E$, then $\mathfrak{J}$ is an $E/\mathfrak{J}$-module and $0 \rightarrow \mathfrak{J} \rightarrow E \rightarrow E/\mathfrak{J} \rightarrow 0$ is an extension of $E/\mathfrak{J}$ by $\mathfrak{J}$.)

The trivial extension of $A$ by $M$ is the exact sequence

$$0 \rightarrow M \xrightarrow{i} M \times A \xrightarrow{\pi} A \rightarrow 0$$

where $i$ is the first coordinate map, where $\pi$ is the second projection and where $M \times A$ is a ring whose underlying additive structure is the direct sum of abelian groups and whose multiplication is given elementwise by $(m, a)(m', a') = (ma' + m'a, aa')$ for all $m, m' \in M$ and all $a, a' \in A$. This extension is denoted by $A \ltimes M$.

Now suppose $A$ is a commutative noetherian local ring with maximal ideal $m$. An $A$-module of finite type $M$ is a canonical module if it has the
three properties:

(i) The natural homomorphism $A \to \text{End}_A(M)$ is a bijection.

(ii) The groups $\text{Ext}_A^i(M, M) = 0$ for all $i > 0$.

(iii) The injective dimension $\text{id}_A M < \infty$. If $B$ is a Gorenstein local ring (see Bass [1]) and if $B \to A$ is a surjection then $A$ is a Cohen-Macaulay ring if and only if $\text{Ext}_B^i(A, B) = 0$ for $i \neq \dim B - \dim A$. If $A$ is Cohen-Macaulay and if $d = \dim B - \dim A$, then $\text{Ext}_B^d(A, B)$ is a canonical $A$-module (see Grothendieck [5] and Sharp [8]). Furthermore $A$ is Gorenstein if and only if $A \cong \text{Ext}_B^d(A, B)$. As a converse to this result Reiten [7] shows that if the Cohen-Macaulay local ring $A$ has a canonical module $M$, then the trivial extension $A \times M$ is a Gorenstein ring. A more precise statement is found in Fossum, Griffith and Reiten [3]: The ring $A \times M$ is a Gorenstein ring if and only if the $A$-module $M$ is a canonical module.

For further properties of canonical modules we refer to Sharp [8], Herzog and Kunz [6], Foxby [4] and Fossum, Griffith and Reiten [3] although the only results which are really needed are:

(a) If $A$ has a canonical module $M$, then $A$ is Cohen-Macaulay.

(b) If $M$ is a canonical $A$-module and if $a \in A$, then $a$ is regular on $M$ if and only if it is regular on $A$.

(c) If $M$ is a canonical $A$-module and if $a$ is a regular element in $A$, then $M/aM$ is a canonical $A/aA$-module. Also we note that when $B \to A$ is a surjective ring homomorphism, when $B$ is Gorenstein and when $A$ is Cohen-Macaulay then the spectral sequence (Cartan and Eilenberg [2]) with $E_2^{pq} = \text{Ext}_A^p(X, \text{Ext}_B^q(A, B))$ and abutment $\text{Ext}_B^n(X, B)$ degenerates to natural isomorphisms $\text{Ext}_A^p(X, \text{Ext}_B^q(A, B)) \cong \text{Ext}_B^{p+q}(X, B)$ for all $A$-modules $X$, for all integers $p \geq 0$ and for $d = \dim B - \dim A$.

(The term canonical module seems to have been introduced by Herzog and Kunz [6]. A more geometric term is module of dualizing differentials, which is very suggestive terminology but also rather cumbersome. Sharp [8] uses the term Gorenstein module of rank one.)

**Arbitrary extensions.** The main result of this paper is the generalization of Reiten's result.

**Theorem.** Suppose $A$ is a local noetherian ring and $M$ is a canonical $A$-module. If $0 \to M \to E \overset{\pi}{\to} A \to 0$ is a commutative extension of $A$ by $M$, then $E$ is a Gorenstein ring.

The proof is based on two lemmas. The first allows a reduction to the Artin case. The second handles the Artin case.

**Lemma 1.** Suppose $A$, $M$ and $E$ are as in the statement of the theorem. An element $e$ in $E$ is regular if and only if $\pi(e)$ is regular in $A$. 
Proof. Since $M$ is a canonical module, an element $\pi(e)$ is regular in $A$ if and only if it is regular on $M$. If $e \in E$ and if $e$ is regular then $e$ is regular on $i(M)$. But the restriction of $e$ to $i(M)$ is the action of $\pi(e)$ on $M$. Hence $\pi(e)$ is regular in $A$.

Suppose, on the other hand, that $\pi(e)$ is regular in $A$. If $e \cdot x = 0$ for some $x \in E$, then $\pi(e) \cdot \pi(x) = 0$. Hence $\pi(x) = 0$, since $\pi(e)$ is regular in $A$. There is then an element $m \in M$ such that $x = i(m)$. Then $ex = i(\pi(e)m)$. Since $\pi(e)$ is regular on $M$ and since $i$ is an injection, the element $m = 0$. Hence $x = 0$. So $e$ is regular in $E$. Q.E.D.

Lemma 2. Suppose $A$ is an Artin ring, that $M$ is an $A$-module with $\text{Ann}_A M = (0)$ and that $E$ is an extension of $A$ by $M$. Then $i(\text{Socle}(M)) = \text{Socle}(E)$.

Proof. Let $n$ be the radical of $E$ and $m$ the radical of $A$ (so that $m = \pi(n)$). Now $e \in \text{Socle}(E)$ if and only if $ne = (0)$. If $ne = (0)$, then $i(M)e = (0)$. But $i(M)e = i(M\pi(e))$. Hence $ne = (0)$ implies $M\pi(e) = (0)$. But $\text{Ann}_A M = (0)$ implies $\pi(e) = 0$. Hence $e \in \text{Socle}(E)$ implies $e \in i(M)$, say $e = i(m)$ for some $m \in M$. Now $n \cdot i(m) = i(m \cdot m)$ so $m \cdot m = (0)$. Hence $m \in \text{Socle}(M)$ and then $e \in i(\text{Socle}(M))$. Clearly $i(\text{Socle}(M)) \subseteq \text{Socle}(E)$. Q.E.D.

Now these two lemmas are used to prove the theorem. The proof is by induction on $\dim A$. We know that $\dim A = \dim E$ since $i(M)$ is nilpotent. By Lemma 1 we can conclude that $\text{depth } A = \text{depth } E$. For suppose $e$ is a regular nonunit in $E$. Then $\pi(e)$ is a regular nonunit in $A$. Multiplication by $e$ induces the commutative diagram with exact rows and columns

\[
\begin{array}{ccccccc}
0 & \to & M & \to & E & \to & A & \to & 0 \\
& & \downarrow^{\pi(e)} & & \downarrow^{e} & & \downarrow^{\pi(e)} & \\
0 & \to & M & \to & E & \to & A & \to & 0.
\end{array}
\]

We conclude that the sequence

\[
0 \to M/\pi(e)M \to E/eE \to A/\pi(e)A \to 0
\]

is exact and is an extension of the local ring $A/\pi(e)A$ by the canonical module $M/\pi(e)M$. Now $\text{depth } E = 1 + \text{depth } E/eE$. By induction depth $E/eE = \text{depth } A/\pi(e)A = -1 + \text{depth } A$. Hence $\text{depth } E = \text{depth } A$.

But we can also use this reduction to show that $E$ is Gorenstein. For $E$ is Gorenstein if and only if $E/eE$ is Gorenstein.

If $\dim A = 0$, then the canonical module $M \cong E(A/m)$, the injective
envelope of the residue class field of $A$. Hence $\text{Ann}_A M = (0)$ since $A \cong \text{End}_A(M)$ and $\text{Socle}(M) \cong A/m$. By Lemma 2 we get $\text{Socle}(E) \cong E/n \cong A/m$. Hence $E$ has a simple socle and is therefore Gorenstein ( Bass [1]).

Suppose, for inductive purposes, that $E'$ is Gorenstein for $\dim E' < \dim E$ whenever $E'$ is an extension of a local ring by a canonical module. Then we conclude that $E$ is Gorenstein since $E/eE$ is Gorenstein. Q.E.D.

**Gorenstein extensions.** Now our attention is directed to the converse problem. Suppose $E$ is an extension of $A$ by $M$, that $E$ is Gorenstein and that $A$ is Cohen-Macaulay. What is the relationship between $M$ and the canonical module $\text{Hom}_K(A, E)$ of $A$? (Note that there are Gorenstein extensions of non-Cohen-Macaulay rings. If $k$ is a field, $X$ and $Y$ indeterminates, then $E = k[[X, Y]]/(X^2)$ is Gorenstein. The ideal generated by the image $XY$ is nilpotent. But $k[[X, Y]]/(X^2, XY)$ is not Cohen-Macaulay.)

Suppose $E$ is a Gorenstein local ring and $\mathfrak{J}$ is an ideal of square zero. Let $A = E/\mathfrak{J}$. We assume that $A$ is Cohen-Macaulay.

Let $B$ be a Gorenstein local ring with $\dim B > 0$. Let $t$ be a regular element in $B$. Then $B/t^n B$ is Gorenstein for all $n \geq 1$. Let $E = B/t^n B$ and let $\mathfrak{J} = t^{n-1} B/t^n B$. Then $E/\mathfrak{J} = B/t^n B$ and $\mathfrak{J}^2 = (0)$. Now $\mathfrak{J}$ is a canonical $E/\mathfrak{J}$-ideal if and only if $n = 2$. Hence it is not the case that $\mathfrak{J}$ is the canonical module.

Let $\Omega = \text{Hom}_K(A, E)$ be the canonical module. Then $\Omega \cong \{ e \in E : e\mathfrak{J} = (0) \}$. Since $\mathfrak{J}^2 = (0)$, we have $\mathfrak{J} \subseteq \Omega$. We get the commutative diagram with exact rows and columns

\[
\begin{array}{ccc}
0 & \rightarrow & \mathfrak{J} & \rightarrow & E & \xrightarrow{\pi} & A & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \Omega & \rightarrow & E & \xrightarrow{\pi'} & \text{Hom}_K(\mathfrak{J}, E) & \rightarrow & 0
\end{array}
\]

The homomorphism $\gamma : A \rightarrow \text{Hom}_K(\mathfrak{J}, E)$ is the composition $A \rightarrow \text{Hom}_K(\mathfrak{J}, \mathfrak{J}) \rightarrow \text{Hom}_K(\mathfrak{J}, E)$. Thus $\ker \gamma = \text{Ann}_A \mathfrak{J}$.

**Lemma 3.** The natural map $A \rightarrow \text{Hom}_K(\mathfrak{J}, \mathfrak{J})$ is a surjection and $\text{Hom}_K(\mathfrak{J}, \mathfrak{J}) \rightarrow \text{Hom}_K(\mathfrak{J}, E)$ is a bijection. Thus $\text{Hom}_K(\mathfrak{J}, \mathfrak{J})$ is identified with both $A/\text{Ann}_A \mathfrak{J}$ and $E/\Omega$.

**Proof.** Since $\mathfrak{J}^2 = (0)$ and since $A$ is Cohen-Macaulay, the group $\text{Ext}_K^i(A, E) = (0)$ for $i > 0$. Hence $\pi'$ is a surjection. Thus $\gamma$ is also a surjection. Since $\text{Hom}_K(\mathfrak{J}, \mathfrak{J}) \rightarrow \text{Hom}_K(\mathfrak{J}, E)$ is an injection, the statements of the lemma follow. Q.E.D.
We identify $\text{Coker}(\mathfrak{I} \to \Omega)$ with $\mathfrak{a}$, the annihilator of $\mathfrak{I}$. The ring $\text{Hom}_E(\mathfrak{I}, \mathfrak{I}) \cong \text{Hom}_A(\mathfrak{I}, \mathfrak{I})$ and is denoted by $A'$. Then the diagram can be displayed as

\[
\begin{array}{ccc}
0 & \to & 0 \\
\downarrow & & \downarrow \\
0 & \to & \mathfrak{I} & \to & E & \to & A & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \Omega & \to & E & \to & A' & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathfrak{a} & \to & 0 & \to & 0 \\
\end{array}
\]

**Lemma 4.** The ring $A'$ is Cohen-Macaulay and $\dim A = \dim A'$.

**Proof.** It is enough to prove that $\text{Ext}^i_E(A', E) = 0$ for $i > 0$. The exact sequence $0 \to \Omega \to E \to A' \to 0$ gives rise to the exact sequence

\[
0 \to \text{Hom}_E(A', E) \to E \to \text{Hom}_E(\Omega, E) \to \text{Ext}^1_E(A', E) \to 0
\]

and the isomorphisms $\text{Ext}^i_E(\Omega, E) \cong \text{Ext}^{i+1}_E(A', E)$ for $i > 0$. Since $A$ is Cohen-Macaulay we get natural isomorphisms $\text{Ext}^i_E(\Omega, E) \cong \text{Ext}^i_A(\Omega, \Omega)$ for all $i$. Hence $\text{Ext}^i_E(\Omega, E) = 0$ for all $i > 0$ while

\[
\text{Hom}_E(\Omega, E) \cong \text{Hom}_A(\Omega, \Omega) \cong A,
\]

since $\Omega$ is a canonical $A$-module. (This follows from the remarks in the introduction.) Thus $E \to \text{Hom}_E(\Omega, E)$ is just $\pi : E \to A$ which is a surjection. So $\text{Ext}^i_E(A', E) = 0$ for all $i > 0$ while the canonical $A'$-module is $\text{Ker} \pi = \mathfrak{I} \cong \text{Hom}_E(A', E) \cong \{ e \in E : e\Omega = 0 \}$. Q.E.D.

We record several other consequences. We retain the hypotheses and notation.

**Proposition.** (a) The sequences

\[
0 \to \mathfrak{I} \to E \xrightarrow{\pi} A \to 0 \quad \text{and} \quad 0 \to \Omega \to E \xrightarrow{\pi'} A' \to 0
\]

are $\text{Hom}_E(-, E)$ dual.

(b) The sequences $0 \to \mathfrak{a} \to A \to A' \to 0$ and $0 \to \mathfrak{I} \to \Omega \to \mathfrak{a} \to 0$ are $\text{Hom}_A(-, \Omega)$ (which is $\text{Hom}_E(-, E)$) dual.

(c) The group $\text{Ext}^i_E(\mathfrak{a}, E) = 0$ for all $i > 0$. 

(d) The $A$-module $\mathcal{Z}$ is a canonical $A'$-module while $\Omega$ is a canonical $A$-module. Thus, in particular, the square zero ideal $\mathcal{Z}$ is a canonical $A$-module if and only if $\text{Ann}_A \mathcal{Z} = (0)$.

(e) We have the following natural isomorphisms:

\[
\begin{align*}
A &= \text{Hom}_A(\Omega, \Omega), \\
A' &= \text{Hom}_A(\mathcal{Z}, \mathcal{Z}) \cong \text{Hom}_E(\mathcal{Z}, E) \cong \text{Hom}_A(\mathcal{Z}, \Omega), \\
\mathcal{Z} &= \text{Hom}_A(A', \Omega) \cong \text{Hom}_E(A', E), \text{ and} \\
\Omega &= \text{Hom}_A(A, \Omega) \cong \text{Hom}_E(A, E). 
\end{align*}
\]

Note that $\Omega$ is a nilpotent ideal in $E$ which has square zero if and only if $\Omega = \mathcal{Z}$.

**Final remarks.** A result which is implicit in this article, and which follows immediately is a slight generalization of the Grothendieck-Bass-Sharp result about Cohen-Macaulay factor rings of Gorenstein rings. We state it here for possible future reference.

**Proposition.** Suppose $A$ is a local Cohen-Macaulay ring with a canonical module $\Omega$. The factor ring $A'$ of $A$ is Cohen-Macaulay if and only if $\text{Ext}_A^i(A', \Omega) = (0)$ for all $i \neq \dim A - \dim A'$. If $A'$ is Cohen-Macaulay, then it has a canonical module which is just the nonzero Ext group. Q.E.D.

It should be mentioned that a nontrivial extension has been exhibited at the beginning of this section. Others will arise from symmetric 2-cocycles $f: A \times A \rightarrow \Omega$ which are not coboundaries (if they exist).

**References**


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