PHYSICAL STATES ON A C*-ALGEBRA

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Abstract. When a locally compact Hausdorff space \(X\) is totally disconnected, any doubly generated subalgebra of (real) \(C_0(X)\) is singly generated.

We have noted that the proof of Lemma 9 of [1] is incomplete. The gap seems to be quite difficult to fill, so Theorem 1 of [1] must be considered as open. We give herein a simple argument which fills that gap for a large class of C*-algebras.

Theorem. Let \(X\) be a totally disconnected locally compact Hausdorff space and \(C_0(X)\) the algebra of all real-valued continuous functions on \(X\) which vanish at infinity. For any \(f, g \in C_0(X)\) there exists \(h \in C_0(X)\) such that \(f\) and \(g\) lie in the closed (sup norm) algebra generated by \(h\). (Hence any quasi-linear [1] functional on \(C_0(X)\) is linear.)

Proof. Since \(X\) is totally disconnected, the Stone-Weierstrass theorem (or a direct construction) shows that the set of all finite linear combinations of idempotents is dense in \(C_0(X)\). For each \(n = 1, 2, \ldots\), there exists a finite family \(\{p^k_i\}_{i=1}^{\infty}\) of idempotents and coefficients \(\{a^n_i\}\) such that
\[
\left\| \sum_{i=1}^{k} a^n_i p^k_i f - f \right\| < 1/n.
\]
Consequently \(\bigcup_{n=1}^{\infty} \{p^k_i\}_{i=1}^{\infty}\) is a countable family, and \(f\) lies in the closed algebra generated by it. Since a similar family exists for \(g\), we may take their union and get a countable family \(\{q^n_i\}_{i=1}^{\infty}\) of idempotents such that both \(f\) and \(g\) lie in the closed subalgebra generated by it. By [2, pp. 293–294] the closed subalgebra generated by \(\{q^n_i\}_{i=1}^{\infty}\) is generated by a single element \(h\), so we are done.

References


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