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SOME PATHOLOGY INVOLVING PSEUDO l-GROUPS AS GROUPS OF DIVISIBILITY

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ABSTRACT. In a partially ordered abelian group G, two elements a and b are *pseudo-disjoint* if $a, b \geqq 0$ and either one is zero, or both are strictly positive and each o-ideal which is maximal with respect to not containing a contains b, and vice versa. G is a *pseudo lattice-group* if every element of G can be written as a difference of pseudo-disjoint elements.

We prove the following theorem: suppose G is an abelian pseudo lattice-group; if there is an $x>0$ and a finite set of pairwise pseudo-disjoint elements $x_1, x_2, \cdots, x_k$ all of which exceed x, and in addition this set is maximal with respect to the above properties, then G is not a group of divisibility.

The main consequence of this result is that every so-called "v-group" $V(\Lambda, R_\lambda)$ for a given partially ordered set Λ, and where R_λ is a subgroup of the additive reals in their usual order, is a group of divisibility only if Λ is a root system, and hence $V(\Lambda, R_\lambda)$ is a lattice-ordered group. We do give examples of pseudo lattice-groups which are not lattice-groups, and yet are groups of divisibility.

Finally, we compute for each integral domain D whose group of divisibility is a lattice-group, the group of divisibility of the polynomial ring $D[x]$ in one variable.

1. **Preliminaries**. All groups in this paper are abelian, and in additive notation unless otherwise indicated. An integral domain here shall be a commutative ring with identity and no zero divisors. If D is an integral domain and K is its quotient field, then the *group of divisibility* of D is the multiplicative group of nonzero elements of K modulo the group $U(D)$ of units of D; in symbols $G(D) \simeq K^*/U(D)$. This group can be given a directed partial order by setting $xU(D) \leqq yU(D)$ if $yx^{-1} \in D$. A (directed) p.o. group G is called a *group of divisibility* if there is an integral domain D such that $G \simeq G(D)$. We can also view this concept in terms of semi-valuations: let K be a field, G be a directed p.o. group, and $v: K^* \to G$

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be a mapping onto G satisfying

(i) $v(xy)=v(x)+v(y)$, for all $x, y \in K^*$;

(ii) $v(-1)=0$;

(iii) $v(x+y) \geqq g$ if $v(x), v(y) \geqq g$, with $x, y \in K^*$ and $g \in G$.

Such a mapping is called a *semivaluation*. Let $D=\{x \in K^* | v(x) \geqq 0\}$; then D is a subring of K, K is its quotient field and $G \simeq G(D)$. Conversely, if D is an integral domain and K is its quotient field, then the canonical mapping $K^* \to G(D)$ is a semivaluation (see [**5**, p. 8]; also [**9**, p. 1148]).

Consequently, G is a group of divisibility if and only if there is a semivaluation onto G.

If G is a totally ordered group (abbreviation o-group), the map v is called a *valuation*, and Krull [**6**, p. 164] demonstrated that every o-group is a group of divisibility. Jaffard [**4**, p. 264] then showed that all lattice-groups (abbreviation l-groups) are groups of divisibility.

In a p.o. group a directed, convex subgroup is called an *o-ideal*. Suppose G is a p.o. group and $0 \leqq a, b \in G$; a and b are *pseudo-disjoint* if either is zero, or both are strictly positive, and every o-ideal which is maximal with respect to not containing a contains b, and vice versa. A *pseudo lattice-group* (abbreviation pseudo l-group) is a p.o. group in which every element can be written as the difference of two pseudo-disjoint elements. For the basic material concerning pseudo l-groups we refer the reader to [**1**] and [**3**]. Conrad shows in [**1**] that in a pseudo l-group G, $0 \leqq a, b \in G$ are pseudo-disjoint if and only if $c \leqq a, b$ implies that $nc \leqq a, b$ for each positive integer n.

For a given partially ordered set Λ, and each $\lambda \in \Lambda$, let R_λ be a subgroup of the additive real numbers equipped with the usual order. Form $V(\Lambda, R_\lambda)$: the subgroup of the cartesian product of the R_λ over Λ consisting of the "vectors" $v=(\cdots, v_\lambda, \cdots)$ whose supports have no infinite ascending chains. $V(\Lambda, R_\lambda)$ becomes a p.o. group by setting $0<v=(\cdots, v, \cdots)$ if $v_\lambda>0$ for each maximal component λ of the support of v. Then $V(\Lambda, R_\lambda)$ is a pseudo l-group (see Theorem 4.8 in [**1**]), and every pseudo l-group may be embedded in some $V(\Lambda, R_\lambda)$ so as to preserve pseudo-disjointness (see 4.11 in [**1**]). It is well known that $V(\Lambda, R_\lambda)$ is an l-group if and only if Λ is a *root system*: $\{\lambda \in \Lambda | \lambda \geqq \lambda_0\}$ is a chain for each $\lambda_0 \in \Lambda$. Finally, two elements $0<v, w \in V(\Lambda, R_\lambda)$ are pseudo-disjoint if and only if no maximal component of the support of v is comparable to one in the support of w [**1**, p. 214].

2. **The main theorem.** We state our main result at the outset.

THEOREM A. *Suppose G is a pseudo l-group, and there is an element $0<x \in G$ and a set $x_1, x_2, \cdots, x_k$ of pairwise pseudo-disjoint elements all of which exceed x, and suppose further that this set is maximal with respect to the above properties. Then G is not a group of divisibility.*

The proof depends on two lemmas, one rather interesting in its own right, the other rather technical.

LEMMA 1. *Suppose G is a pseudo l-group, and v is a semivaluation from a field K upon G. If $0<a, b \in G$ are pseudo-disjoint and $0<c<a, b$, then there is an element $0<g \in G$, pseudo-disjoint to a and b, with $c<g$.*

PROOF. Let $v(x)=a$, $v(y)=b$ and $g=v(x+y)$. If $c \leqq a, g$ then $c \leqq v(-x)$, so that $b=v(y)=v(x+y-x) \geqq c$. But a and b are pseudo-disjoint and hence $nc \leqq a, b$, for any positive integer n. Again using one of the defining properties of semivaluations $nc \leqq g$. Conclusion: a and g are pseudo-disjoint; likewise b and g are pseudo-disjoint. It is clear that if $c<a, b$ then $c<g$; in particular $g>0$.

If G is a pseudo l-group and $0 \neq x \in G$ we call an o-ideal M of G which is maximal with respect to not containing x a *value* of x. In this language then, a is pseudo-disjoint to b if and only if every value of a contains b, and vice versa.

LEMMA 2. *Suppose G is a pseudo l-group and $0<a \in G$, $0<b_i \in G$ $(i=1, \cdots, k)$. Assume further that the b_i are pairwise pseudo-disjoint, while a is pseudo-disjoint to $b_1+b_2+\cdots+b_k$. Then a is pseudo-disjoint to each b_i.*

PROOF. Let M be a value of a; then by our assumption $b_1+b_2+\cdots+b_k$ is in M, and so by convexity each $b_i \in M$. On the other hand if N is a value of b_i, each $b_j \in N$, for $j \neq i$; this makes N a value of $b_1+b_2+\cdots+b_k$, and hence $a \in N$. It follows then that each b_i is pseudo-disjoint to a.

PROOF OF THEOREM A. Suppose G is a pseudo l-group, $0<x \in G$ and $x_1, x_2, \cdots, x_k$ is a maximal, pairwise pseudo-disjoint set of elements of G exceeding x. Relabel $x_1=a$ and $b=x_2+x_3+\cdots+x_k$; then a and b are pseudo-disjoint.

If G is a group of divisibility as well, there is semivaluation v from a field K onto G. By Lemma 1 we may find $0<g \in G$ pseudo-disjoint to both a and b, such that $x<g$. By Lemma 2 g is pseudo-disjoint to each x_i $(i=1, \cdots, k)$; this contradicts the maximality of the set $x_1, x_2, \cdots, x_k$ over x.

This proves the theorem.

Our first corollary concerns v-groups.

THEOREM B. *Let Λ be a partially ordered set, R_λ be an ordered subgroup of the reals for each $\lambda \in \Lambda$; set $V=V(\Lambda, R_\lambda)$. If V is a group of divisibility then Λ is a root system and hence V is an l-group.*

PROOF. If Λ is not a root system there exists a $\nu \in \Lambda$ with pairwise incomparable elements above ν in Λ. Let $\{\lambda_i | i \in I\}$ be a set of mutually incomparable elements of Λ all of which exceed ν, and suppose $\{\lambda_i | i \in I\}$ is also maximal with respect to these properties. Fix $j \in I$ and define $a, b \in V$ as follows:

$$\begin{aligned} a_\lambda &= 1, \quad \text{if } \lambda = \lambda_j, & b_\lambda &= 1, \quad \text{if } \lambda = \lambda_i,\ i \neq j, \\ &= 0, \quad \text{otherwise}; & &= 0, \quad \text{otherwise.}^1 \end{aligned}$$

Clearly $0 < a,\ b \in V$ and a is pseudo-disjoint to b; moreover the pair $\{a, b\}$ satisfies the conditions of Theorem A relative to, say, $x \in V$, where

$$\begin{aligned} x_\lambda &= 1 \quad \text{if } \lambda = \nu; \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

By the theorem we obtain a contradiction: for if there is an element $0 < g \in G$, pseudo-disjoint to both a and b which exceeds x, then we contradict the maximality of the set $\{\lambda_i | i \in I\}$ over ν. Thus V cannot be a group of divisibility unless Λ is a root system.

If G is a pseudo l-group and $0 < u \in G$ has the property that no strictly positive element is pseudo-disjoint to u, we call u a *weak order unit*.

COROLLARY 1. *Suppose the pseudo l-group G has a weak order unit u which can be written as the sum of a pair of pseudo-disjoint elements which are not disjoint. Then G is not a group of divisibility.*

PROOF. Write $u = a + b$ with $a, b > 0$ in G as prescribed in the statement of the corollary, and suppose $0 < c < a, b$. Then $\{a, b\}$ is a maximal pseudo-disjoint set over c, and Theorem A applies.

Let G be a p.o. group and A be an o-ideal of G. We call G a *lex-extension* of A (by G/A) if for each $0 < a \in A$ and $0 < g \in G \backslash A$, $g > a$. G is a *direct* lex-extension of A if A is a direct summand: equivalently, $G = B \oplus A$ and $0 \leqq g = (b, a)$ if and only if $b > 0$, or $b = 0$ and $a \geqq 0$. We then write $G = B \vec{\times} A$. If A and B are l-groups then $G = B \vec{\times} A$ is a pseudo l-group [**3**], and under these assumptions G is an l-group if and only if $A = 0$ or B is an o-group.

Call a weak order unit u in an l-group B *decomposable* if u can be written as a sum of pairwise disjoint, strictly positive elements of B.

COROLLARY 2. *Let $A \neq 0$ and B be l-groups, and suppose that B has a decomposable weak unit. Then $G = B \vec{\times} A$ is not a group of divisibility.*

We compare our last corollary with Ohm's theorem 5.3 in [**8**]. Consider his condition labeled (5.1): there exist $b_1, b_2 \in B$ such that b_1 and b_2 are

[1] We may assume without loss of generality that the number 1 is in each R_λ.

incomparable, and a subdirect representation of B as a subdirect product of o-groups B_i $(i \in I)$ by an l-isomorphism σ such that $b_1\sigma_i \neq b_2\sigma_i$, for all $i \in I$. It is equivalent to the existence of a decomposable weak order unit in B.

To see this note that if Ohm's (5.1) holds for an l-group B, and b_1 and b_2 are as specified above, then if we set $u=(b_1-b_2)\vee 0+(b_2-b_1)\vee 0$, u is a decomposable weak order unit. For $u\sigma_i=(b_1-b_2)\sigma_i\vee 0+(b_2-b_1)\sigma_i\vee 0$, and so $u\sigma_i=(b_1-b_2)\sigma_i$ or $(b_2-b_1)\sigma_i$, either of which is >0. Hence u is a weak order unit, and it is clearly decomposable.

Conversely, suppose B has a decomposable weak order unit u, and $u=a+b$, with $0<a$, $b \in B$ and $a\wedge b=0$. If a minimal prime subgroup N of B contains u then by the minimality of N there exists an element $0<x \in B\backslash N$ such that $x\wedge u=0$, a contradiction. Consider then the family $\{N_\lambda | \lambda \in \Lambda\}$ of minimal prime subgroups of B; let $B_\lambda=B/N_\lambda$ and $\sigma: B\to \Pi B_\lambda$ be the induced l-embedding. Each B_λ is an o-group and $u\sigma_\lambda>0$, for each $\lambda \in \Lambda$. Let $b_1=a-b$ and $b_2=0$; then this pair satisfies Ohm's condition relative to the mapping σ. (We refer the reader to [**2**, pp. 1.14–1.15 and pp. 2.13–2.14].)

His Theorem 5.3 is somewhat more general than Corollary 2 in view of the fact that we assume A to be an l-group, whereas he does not.

Following Corollary 3.3 in [**8**] Ohm remarks that if one takes the polynomial ring $k[x, y]$ in two indeterminates over the field k, and localizes by the ideal generated by x and y, one obtains a local ring whose group of divisibility is a cardinal sum of copies of Z, the integers in their usual order; the number of copies of Z is at least 2 since the local ring is not a valuation ring. If G is then the group of divisibility of a domain D whose quotient field is k, Corollary 3.3 in [**8**] shows that the direct lex-extension of G by this cardinal sum of integers is again a group of divisibility. If G is an l-group such a lex-extension is a pseudo l-group which is not an l-group, providing a large class of examples of such pseudo l-groups which are groups of divisibility.[2] In view of the observation in §1 that every pseudo l-group can be embedded in a reasonably "nice" way in a v-group, the examples here contrasted with Theorem B leave a rather monstrous question mark as to the nature of groups of divisibility, not only in the context of pseudo l-groups, but in general as well.

3. **Polynomial rings and Gauss' lemma.** We conclude this note with a result that calculates for an integral domain D whose group of divisibility is an l-group, the group of divisibility of its polynomial ring $D[x]$ in one variable. Curiously, an analogue of the classical Gauss lemma for

[2] In view of Theorem A there are infinitely many copies of Z in these cardinal sums.

polynomials crops up at a rather crucial juncture. First, a general preliminary remark:

PROPOSITION. *Let D be an integral domain, G be its group of divisibility; then $G(D[x])$ is a direct extension of G by a cardinal sum of copies of Z.*

PROOF. Let k be the quotient field of D. We note here that the group of units $U(D)$ of D is also the group of units of $D[x]$. Further $D[x]$ and $k[x]$ have same quotient field, namely $k(x)$, the field of rational functions in x with coefficients in k. Finally, the group of units of $k[x]$ is k^*. Thus

$$G = k^*/U(D), \qquad G(D[x]) = k(x)^*/U(D), \quad \text{and} \quad G(k[x]) = k(x)^*/k^*,$$

and the latter is a cardinal sum of integers; see [7, Theorem 4.3]. Clearly, the inclusion of G in $G(D[x])$ is a convex order embedding, and the canonical epimorphism $G(D[x])\to G(k[x])$ is an o-epimorphism. Hence $G(D[x])/G\simeq G(k[x])$; since $G(k[x])$ is abstractly a free abelian group, the extension is direct.

Now suppose $G=G(D)$ is an l-group; then D has the following properties:

(1) any finite set of nonzero elements of D has a greatest common divisor, and

(2) if d divides ab $(a, b, d \in D)$ then $d=xy$ where x divides a and y divides b. This is so because G, being an l-group, satisfies the *Riesz interpolation property*: if $0\leqq a_1$, $a_2 \in G$ and $0\leqq b \in G$, then $b\leqq a_1+a_2$ implies that $b=b_1+b_2$, with $0\leqq b_i\leqq a_i$ $(i=1, 2)$.

Call a polynomial $p(x)$ in $D[x]$ *primitive* if the greatest common divisor of the coefficients of $p(x)$ is a unit of D. If G is an l-group any polynomial $g(x) \in D[x]$ can be written uniquely (up to units) as $g(x)=d\cdot g_0(x)$, where $g_0(x)$ is primitive and d is the greatest common divisor of the coefficients of $g(x)$.

The following is a crucial lemma.

LEMMA 3 (GAUSS' LEMMA). *If the group of divisibility G of an integral domain D satisfies the Riesz interpolation property, the product of two primitive polynomials in $D[x]$ is primitive.*

PROOF. Let $p(x)=a_0+a_1x+\cdots+a_mx^m$ and $q(x)=b_0+b_1x+\cdots+b_nx^n$ be primitive polynomials, and $p(x)q(x)=c_0+c_1x+\cdots+c_{m+n}x^{m+n}$. Suppose $d \in D$ divides all c_k, and is not a unit. Let i_0 (j_0) be the first index such that d fails to divide a_{i_0} (b_{j_0}); set $k_0=i_0+j_0$. Then d divides $c_{k_0}=a_0b_{k_0}+\cdots+a_{i_0}b_{j_0}+\cdots+a_{k_0}b_0$, and so d divides $a_{i_0}b_{j_0}$. Since G satisfies the Riesz interpolation property $d=x_0y_0$ where x_0 (y_0) divides a_{i_0} (b_{j_0}). Now x_0 divides each c_k, each a_i for $i=0, 1, \cdots, i_0$ and each b_j for $j=0, 1, \cdots, j_0-1$.

By induction, x_0 is a unit and so d divides b_{j_0}, which is a contradiction. We conclude that $p(x)q(x)$ is primitive, and the lemma is proved.

THEOREM C. *If the group of divisibility G of the integral domain D is an l-group, then $G(D[x])$ is a cardinal sum of G with a cardinal sum of copies of Z; in particular $G(D[x])$ is an l-group.*

PROOF. Recall that a *saturated multiplicative system* of an integral domain is a subset of nonzero elements, closed under multiplication, which contains along with an element d all the divisors of d. Mott (see [7, Theorem 5.1]) showed that there is a natural isomorphism between the lattice of saturated multiplicative systems of an integral domain and the o-ideals of its group of divisibility.

Lemma 3 says that the subset S of primitive polynomials in $D[x]$ is multiplicative; it is clearly saturated. Also, the nonzero elements of D form a multiplicative system in $D[x]$ which is saturated; denote this subset by D^*. Since G is an l-group we may write every nonzero polynomial $f(x)$ as a product of an element from D^* and an element of S; evidently $S \cap D^* = U(D)$. By Mott's Theorem (and the logical extension thereof) there exist o-ideals A and B of $G(D[x])$ such that $G(D[x])$ is the cardinal sum of A and B; if A corresponds to D^* then clearly $A \simeq G$, and it is immediate that B (corresponding to S) is isomorphic to $G(k[x])$. This concludes the proof of Theorem C.

We offer the following remark in the way of a converse of Theorem C. Let D be an integral domain and G be its group of divisibility. Without any further assumptions G is an o-ideal of $G(D[x])$; so suppose it splits off cardinally. Then $G(D[x]) = G \uplus M$, where M is an o-ideal of $G(D[x])$; using Mott's correspondence again we come up with a saturated multiplicative system T in $D[x]$ having the properties that (1) $D^* \cap T = U(D)$ and (2) every nonzero polynomial $f(x)$ can be written (uniquely up to units) as the product of an element of D^* and one from T. Now let S be the *set* of primitive polynomials; clearly $S \subseteq T$, and if $p(x) \in T$ but is not primitive, then write $p(x) = d \cdot q(x)$, and pick d to be a nonunit of D. Since T is saturated $q(x) \in T$, but this violates the uniqueness of such expressions. Hence $T = S$.

Moreover pick $0 \neq a, b \in D$ and consider $f(x) = a + bx$; by writing $f(x)$ as a product of an element from D^* and an element from S we locate the greatest common divisor of a and b. We can therefore make the following conclusion.

THEOREM D. *Let G be the group of divisibility of the integral domain D; let $H = G(D[x])$. If H is the cardinal sum of G and $G(k[x])$ then*

(1) *any finite set of nonzero elements of D has a greatest common*

divisor, and

(2) *the subset S of primitive polynomials over D is a saturated multiplicative system.*

If G satisfies the Riesz interpolation property it is an l-group.

Finally, in view of Theorem C *conditions* (1) *and* (2) *are sufficient to insure that G split as a cardinal summand of H.*

In closing we pose one of many questions that arise naturally here: if $G=G(D)$ satisfies the Riesz interpolation property, then does $G(D[x])$?

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