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MINIMAL PRIMES OF IDEALS AND INTEGRAL RING EXTENSIONS

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ABSTRACT. It is shown that if R is a commutative ring with identity having the property that ideals in R have only a finite number of minimal primes, then a finite R-algebra again has this property. It is also shown that an almost finite integral extension of a noetherian integral domain has noetherian prime spectrum.

If $\mathfrak{a}$ is an ideal in a ring R (tacitly assumed to be commutative with identity) and P is a prime ideal of R containing $\mathfrak{a}$, then P is called a *minimal prime* of $\mathfrak{a}$ if there is no prime ideal of R containing $\mathfrak{a}$ and properly contained in P. The ring R is said to have FC (for finite components) if each ideal of R has only a finite number of minimal primes.

I am indebted to Professor Nagata for suggestions which helped me in obtaining the proofs of the theorems in this article, and I would like to thank him for his generous help.

The special case in Theorem 1 where R is an integrally closed domain was proved in [**1**, Corollary 11, p. 577].

THEOREM 1. *If R is a ring with FC and R' is a finite R-algebra, then R' has FC.*

PROOF. It clearly suffices to consider the case where $R'=R[\xi]$ is a simple R-algebra.[2] Moreover, since FC is preserved under homomorphic image, we may assume $R[\xi]=R[X]/(f(X))$, where $f(X)$ is a monic polynomial. Suppose there exists an ideal $\mathfrak{a}'$ in R' having an infinite number of minimal primes, say $\{P'_\alpha\}$. We may assume that $\mathfrak{a}'=\bigcap_\alpha P'_\alpha$. Since R has FC, $\mathfrak{a}'\cap R=\mathfrak{a}$ has only finitely many minimal primes, so there must exist a minimal prime P_1 of $\mathfrak{a}$ such that P_1 is contained in infinitely many of the P'_α. Since $R'=R[X]/(f(X))$ is a finite free R-module every minimal

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[2] It is not true in general that R having FC implies the polynomial ring $R[X]$ has FC [**3**, Example 2.9, p. 635].

prime of P_1R' lies over P_1 in R and P_1R' has only finitely many minimal primes. It follows that some minimal prime P_1' of P_1R' is contained in infinitely many of the P_α'. Let $\mathfrak{a}_1' = \bigcap \{P_\alpha' \mid P_1' \subset P_\alpha'\}$. The fact that P_1' is contained in infinitely many of the P_α' implies that $\mathfrak{a}'$ is not contained in P_1' and P_1' is properly contained in $\mathfrak{a}_1'$. Thus $\mathfrak{a}_1'$ has infinitely many minimal primes and $\mathfrak{a}_1' \cap R = \mathfrak{a}_1$ properly contains P_1. We note also that some minimal prime of P_1R' is in the set $\{P_\alpha'\}$. For P_1 occurs in a representation of $\mathfrak{a} = \mathfrak{a}' \cap R$ as a finite irredundant intersection of prime ideals. Hence if $\mathfrak{b}' = \bigcap \{P_\alpha' \mid P_1 \subset P_\alpha'\}$, then $\mathfrak{b}' \cap R = P_1$. Since P_1R' has only finitely many minimal primes, it follows that some P_α' is a minimal prime of P_1R'. Proceeding now in this manner with $\mathfrak{a}_1'$, we can construct for any positive integer n, a chain $P_1 < \cdots < P_n$ of prime ideals in R such that each P_i is the contraction of some prime in the set $\{P_\alpha'\}$. By choosing $n > \deg f = m$, we show that this leads to a contradiction. We may assume that $P_1 = (0)$, and hence that R is an integral domain. Let T be the integral closure of R in an algebraic closure of the quotient field of R and let $(0) = Q_1 < \cdots < Q_n$ be primes of T such that $Q_i \cap R = P_i$. Let $T[\xi] = T[X]/(f(X)) = T \otimes_R R[\xi]$ and identify T and $R[\xi]$ as subrings of $T \otimes_R R[\xi]$, $T = T \otimes 1$ and $R[\xi] = 1 \otimes R[\xi]$. In $T[X]$ we have $f(X) = (X - \xi_1) \cdots (X - \xi_m)$. Let Q_{ij}' denote the prime in $T[\xi]$ lying over Q_i in T and corresponding to the root ξ_j of $f(X)$. Of course not all the Q_{ij}', $1 \leqq j \leqq m$, need be distinct, but we have $Q_{1j}' < Q_{2j}' < \cdots < Q_{nj}'$. Moreover, if $Q_{ij}' \cap R[\xi] = P_{ij}'$, then $\{P_{ij}'\}_{j=1}^m$ is the set of minimal primes of $P_iR[\xi]$ and $P_{1j}' < P_{2j}' < \cdots < P_{nj}'$. But by our construction, for each i, some $P_{ij}' \in \{P_\alpha'\}$. Since there are no containment relations among the elements of $\{P_\alpha'\}$, this implies that $n \leqq m$ and completes the proof. Q.E.D.

A ring R is said to have *noetherian spectrum* if the radical ideals in R satisfy the ascending chain condition. Conditions equivalent to R having noetherian spectrum are that R have FC and satisfy the ascending chain condition on prime ideals.

If $R \subset R'$ are integral domains, then R' is said to be *almost finite* over R if R' is integral over R and if the quotient field of R' is a finite algebraic extension of the quotient field of R [**2**, p. 30].

THEOREM 2. *If R is a noetherian integral domain and R' is an almost finite extension of R, then R' has noetherian spectrum.*

PROOF. It will suffice to consider the case when R' is integrally closed, for if the integral closure of R' has noetherian spectrum then so does R'. We first show that, for any ideal $\mathfrak{a}$ in R, $\mathfrak{a}R'$ has only a finite number of minimal primes. We proceed by induction on the number of generators for $\mathfrak{a}$. Principal ideals in R' have only finitely many minimal primes, for R' is the derived normal ring of a noetherian domain and hence is a

Krull domain [2, p. 118]. If $\mathfrak{a}=(x_1, \cdots, x_n)$, let $P'_1, \cdots, P'_m$ be the minimal primes of x_1R'. Then R'/P'_i is an almost finite extension of $R/(P'_i \cap R)$ [2, p. 118], and by our induction assumption $\mathfrak{a}(R'/P'_i)$ has only finitely many minimal primes. Since every minimal prime of $\mathfrak{a}R'$ contains at least one of the P'_i, it follows that $\mathfrak{a}R'$ has only finitely many minimal primes. To complete the proof of the theorem, we can now proceed as as in Theorem 1, viz. if $\mathfrak{a}'$, were an ideal in R' having infinitely many minimal primes then we could construct in R an infinite strictly ascending chain $P_1 < P_2 < \cdots$ of prime ideals. This would of course contradict the fact that R is noetherian.

COROLLARY. *If R is a noetherian integral domain, then the derived normal ring of R has noetherian spectrum.*

In connection with properties of the derived normal ring of a noetherian integral domain, we would like to ask the following.

Question. If R is a noetherian integral domain with integral closure $\bar{R}$, must it follow that maximal ideals in $\bar{R}$ are finitely generated?

In trying to show this to be true, a simple induction argument on the Krull dimension of R runs into the difficulty that there can exist between a 2-dimensional noetherian domain A and the integral closure of A a ring B having a non-finitely-generated maximal ideal. To get an example illustrating this one can use the following construction suggested to me by Kaplansky. Let T be a 1-dimensional local (noetherian) domain such that the integral closure of T is not a finite T-module. Let $T < T_1 < T_2 \cdots$ be a strictly ascending chain of finite T-algebras between T and the integral closure of T. Let X be an indeterminate and let $A = T[X]$, and $B = T + T_1X + T_2X^2 + \cdots$. Then B is a ring between A and the integral closure of A and if m is the maximal ideal in T, then $m + T_1X + T_2X^2 \cdots$ is a non-finitely-generated maximal ideal in B.

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