CLASSIFICATION OF BOUNDED SOLUTIONS
OF A LINEAR NONHOMOGENEOUS
DIFFERENTIAL EQUATION

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ABSTRACT. An elementary criterion, depending only upon the
initial data of a solution, is formulated to determine the bounded-
ness of solutions of a nonhomogeneous linear system of ordinary
differential equations. The associated homogeneous linear differ-
ential equation is required to be either conditionally stable or
conditionally asymptotically stable.

1. Introduction. Substantive information for linear perturbation
problems includes a knowledge of the behavior of the solutions of the non-
homogeneous linear differential equation

\begin{equation}
    x' = A(t)x + f(t)
\end{equation}

whenever the associated homogeneous system

\begin{equation}
    y' = A(t)y
\end{equation}

possesses a prescribed property. In this note, we assume that the homo-
genous equation (2) possesses a certain conditional stability property;
then, a classification of the bounded solutions of (1) is obtained. The
boundedness criterion is phrased solely in terms of the initial value of the
solution.

2. Hypotheses for (1) and (2). In equations (1) and (2), \( x, y \) are el-
ements in an \( n \)-dimensional vector space \( X \); \( A(t) \) is a continuous \( n \times n \)
matrix defined on \( R=\langle -\infty, \infty \rangle \); and, \( f \in C[R, X] \). Let \( Y(t) \) denote the
fundamental matrix of (1) that satisfies the condition \( Y(0) = I_n \), \( I_n \) is the
\( n \times n \) identity matrix. The symbol \( |\cdot| \) denotes a norm on \( X \) as well as a
corresponding (consistent) matrix norm.

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3. The conditional asymptotic stability case. A fundamental requirement for (2) in this section is

(H.1) Let there exist supplementary projections $P_0, P_{-1}, P_1, P_\infty$ and constants $K, q$ with $K > 0$ and $1 \leq q < \infty$ such that

$$\left[ \int_{-\infty}^{t} |Y(t)P_{-1}Y^{-1}(s)|^q \, ds \right]^{1/q} + \left[ \int_{0}^{t} |Y(t)P_0Y^{-1}(s)|^q \, ds \right]^{1/q} + \left[ \int_{t}^{\infty} |Y(t)P_1Y^{-1}(s)|^q \, ds \right]^{1/q} \leq K, \quad t \in \mathbb{R};$$

$$\left[ \int_{t}^{\infty} |Y(t)P_{\infty}Y^{-1}(s)|^q \, ds \right]^{1/q} \leq K, \quad t \in \mathbb{R}_+ \equiv [0, \infty);$$

$$\left[ \int_{-\infty}^{t} |Y(t)P_0Y^{-1}(s)|^q \, ds \right]^{1/q} \leq K, \quad t \in \mathbb{R}_- \equiv (-\infty, 0].$$

A complementary requirement on (1) is

(H.2) The nonhomogeneous term $f$ is in $L^p(\mathbb{R})$ where $1/p + 1/q = 1$. The norm of $f \in L^p(\mathbb{R})$ is denoted by $|f|_p$.

**Theorem 1.** Let conditions (H.1) and (H.2) be satisfied for equations (1) and (2). Then, the following conclusions hold:

(i) A solution $x = x(t)$ of (1) is bounded on $\mathbb{R}_+$ if and only if

$$P_1x(0) = -\int_{0}^{\infty} P_1Y^{-1}(s)f(s) \, ds; \quad \text{and}$$

(3)

$$P_\infty x(0) = -\int_{-\infty}^{0} P_\infty Y^{-1}(s)f(s) \, ds.$$

(ii) A solution $x = x(t)$ of (2) is bounded on $\mathbb{R}_-$ if and only if

$$P_{-1}x(0) = \int_{-\infty}^{0} P_{-1}Y^{-1}(s)f(s) \, ds; \quad \text{and}$$

(4)

$$P_\infty x(0) = \int_{-\infty}^{0} P_\infty Y^{-1}(s)f(s) \, ds.$$

(iii) A solution $x = x(t)$ of (2) is bounded on $\mathbb{R}$ if and only if both (3) and (4) hold.

**Proof.** The hypothesis (H.1) used in conjunction with the Lemmas of [4] (also see R. Conti [1] and W. A. Coppel [2, pp. 68, 74]) imply

$$\lim_{|t| \to \infty} |Y(t)P_0| = 0; \quad \text{(5)}$$

$$\lim_{t \to \infty} |Y(t)P_{-1}| = 0 \quad \text{and}$$

$$\limsup_{t \to -\infty} |Y(t)P_{-1}\xi| = \infty \quad \text{provided } P_{-1}\xi \neq 0; \quad \text{(6)}$$
\[ \lim_{t \to -\infty} |Y(t)P|_1 = 0 \quad \text{and} \quad \lim_{t \to -\infty} \sup |Y(t)P_\infty| = \infty \quad \text{provided } P_\infty \neq 0; \]

\[ \lim_{|t| \to \infty} |Y(t)P|_\infty = \infty \quad \text{provided } P_\infty \neq 0. \]

It follows from (5), (6), and (7) that there is a constant \( M > 0 \) such that

\[ |Y(t)P_{-1}| + |Y(t)P_0| \leq M, \quad t \in R_+; \quad \text{and} \]

\[ |Y(t)P_1| + |Y(t)P_0| \leq M, \quad t \in R_. \]

Proceeding formally, we use the variation of parameters formula to obtain for a solution \( x \) of (1) valid on \( R_+ \),

\[ x(t) = Y(t)x(0) + \int_0^t Y(t)Y^{-1}(s)f(s) \, ds \]

\[ = Y(t)[P_0 + P_{-1} + P_1 + P_\infty]x(0) + \int_{-\infty}^t Y(t)P_{-1}^{-1}(s)f(s) \, ds \]

\[ + \int_0^t Y(t)P_0 Y^{-1}(s)f(s) \, ds - \int_t^\infty Y(t)[P_1 + P_\infty]Y^{-1}(s)f(s) \, ds \]

\[ + \int_0^\infty Y(t)[P_1 + P_\infty]Y^{-1}(s)f(s) \, ds \]

\[ - \int_{-\infty}^0 Y(t)P_{-1}^{-1}(s)f(s) \, ds, \quad t \in R_. \]

For \( t \in R_- \), we have

\[ x(t) = Y(t)[P_0 + P_{-1} + P_1 + P_\infty]x(0) \]

\[ + \int_{-\infty}^t Y(t)[P_{-1} + P_\infty]Y^{-1}(s)f(s) \, ds \]

\[ + \int_0^t Y(t)P_0 Y^{-1}(s)f(s) \, ds - \int_t^\infty Y(t)P_1 Y^{-1}(s)f(s) \, ds \]

\[ - \int_{-\infty}^0 Y(t)[P_{-1} + P_\infty]Y^{-1}(s)f(s) \, ds + \int_0^\infty Y(t)P_1 Y^{-1}(s)f(s) \, ds. \]

It is a direct consequence of Hölder's inequality, hypotheses (H.1) and (H.2) that each of the above integrals, with the exception of the last ones, in (9) and (10) is well defined. To verify that the last integrals are also well defined, we evaluate an equality in (H.1) at \( t = 0 \) to obtain

\[ \left[ \int_{-\infty}^0 |P_{-1}Y^{-1}(s)|^q \, ds \right]^{1/q} + \left[ \int_0^\infty |P_1 Y^{-1}(s)|^q \, ds \right]^{1/q} \leq K. \]
From (11) and (H.2), we have

\[ \left| \int_{-\infty}^{0} Y(t)P_{-1}Y^{-1}(s)f(s)\,ds \right| \]

\[ \leq |Y(t)P_{-1}| \left[ \int_{-\infty}^{0} |P_{-1}Y^{-1}(s)|^{q}\,ds \right]^{1/q} |f|_{p} \leq MK |f|_{p}. \]

The same bound also holds for the last integral in (10); hence, \( x=x(t) \) is well defined. A direct calculation shows that the function \( x=x(t) \) with domain \( R \) defined by (9) for \( t \in R_{+} \) and by (10) for \( t \in R_{-} \) is the solution of (1) that passes through the initial position \( x(0) \).

We first verify that assertion (i) holds. Since \( x \) is given by (9) for \( t \in R_{+} \), an application of Hölder’s inequality, conditions (H.1), (H.2) and inequality (12) leads to

\[ x(t) - Y(t)[P_{1} + P_{\infty}] \left[ x(0) + \int_{0}^{\infty} Y^{-1}(s)f(s)\,ds \right] \]

\[ \leq M \left| [P_{1} + P_{\alpha}]x(0) \right| + (2 + M)K |f|_{p}, \quad t \in R_{+}. \]

From (13), we note that the statement—\( x(t) \) is bounded on \( R_{+} \)—is equivalent to the statement—

\[ Y(t)[P_{1} + P_{\infty}] \left[ x(0) + \int_{0}^{\infty} Y^{-1}(s)f(s)\,ds \right] \]

is bounded on \( R_{+} \). Equations (7) and (8) show that the expression in (14) is bounded on \( R_{+} \) if and only if (3) holds; this proves (i).

Conclusion (ii) is verified in an analogous fashion using equation (10). The analogue of inequality (13) is

\[ x(t) - Y(t)[P_{-1} + P_{\infty}] \left[ x(0) - \int_{-\infty}^{0} Y^{-1}(s)f(s)\,ds \right] \]

\[ \leq M \left| [P_{1} + P_{\alpha}]x(0) \right| + (2 + M)K |f|_{p}, \quad t \in R_{-}. \]

This inequality coupled with equations (6) and (8) leads to (ii).

The assertion (iii) is an immediate consequence of (i) and (ii).

Remark 1. As a corollary to (iii), we find that a necessary condition for (2) to possess a solution that is bounded on \( R \) is that the forcing function \( f \) satisfy the equation

\[ \int_{-\infty}^{\infty} P_{\infty} Y^{-1}(s)f(s)\,ds = 0. \]

If it is known that (2) has a bounded solution for every \( f \in L^{p} \) (see [1] for this type of result on \( R_{+} \)) then, from (15), we have \( P_{\infty}=0 \). This extends a recent result of D. L. Lovelady [6] who considered the case \( p=\infty \).
Remark 2. If the conditional asymptotic stability hypothesis (H.1) is replaced by the corresponding conditional stability analogue then it is no longer necessary, that an $L^\infty$-decomposition of the solution space is effected. However, Theorem 1 still holds if it is assumed that (H.1) and (H.2) are replaced by the corresponding requirements in $L^\infty$ and $L^1$ respectively and, in addition, that

\[
\limsup_{t \to -\infty} |Y(t)P_{-1} \xi| = \infty \quad \text{provided } P_{-1} \xi \neq 0,
\]
\[
\limsup_{t \to \infty} |Y(t)P_1 \xi| = \infty \quad \text{provided } P_1 \xi \neq 0, \quad \text{and}
\]
\[
\limsup_{|t| \to \infty} |Y(t)P_\infty \xi| = \infty \quad \text{provided } P_\infty \xi \neq 0.
\]

With these assumptions, the proof parallels that of the case $1 \leq q < \infty$.

4. Concluding remarks. The projection $P_\infty$ has not been frequently utilized in connection with perturbation problems. Most research efforts in this direction have been devoted to finding bounded solutions of differential equations; hence, $P_\infty$ has been purposefully neglected.

In [5], a discussion of the unbounded solutions of a functional perturbation problem is presented under conditions where $P_\infty$ need not be zero. A different integral equation representation and hypothesis on $P_\infty$ is used there.

Information about the behavior of the solutions of (2) is necessary before the above theorems can be applied. J. Macki and J. Muldowney [7] discuss the bounded and unbounded solutions of (2). Other information about the bounded solutions of (2) can be found in [3].

We conclude this note with a simple but illustrative example. Let the vector $y$ be given by $y = \text{col}(y_1, y_2, y_3, y_4)$ and the matrix $A$ be given by $A(t) = \text{diag}(-2t, -1, 1, 2t)$. The fundamental matrix $Y$ of this system (2) with $Y(0) = I_4$ is $Y(t) = \text{diag}(e^{-t^2}, e^{-t}, e^t, e^{t^2})$. The projections $P_i$ of condition (H.1) are taken to be

\[
P_{-1} = \text{diag}(0, 1, 0, 0); \quad P_0 = \text{diag}(1, 0, 0, 0);
\]
\[
P_1 = \text{diag}(0, 0, 1, 0); \quad P_\infty = \text{diag}(0, 0, 0, 1).
\]

The matrices $Y(t)P_i Y^{-1}(s)$ are given by

\[
Y(t)P_{-1} Y^{-1}(s) = \text{diag}(0, \exp[s - t], 0, 0);
\]
\[
Y(t)P_0 Y^{-1}(s) = \text{diag}(\exp[s^2 - t^2], 0, 0, 0);
\]
\[
Y(t)P_1 Y^{-1}(s) = \text{diag}(0, 0, \exp[t - s], 0);
\]
\[
Y(t)P_\infty Y^{-1}(s) = \text{diag}(0, 0, 0, \exp[t^2 - s^2]).
\]
The hypothesis (H.1) is satisfied for $q=1$. Thus, if $f=\text{col}(f_1, f_2, f_3, f_4)$ is bounded and continuous on $R$, then Theorem 1 leads to the following criteria for the determination of the boundedness of solutions of (1).

A solution $x(t)=\text{col}(x_1(t), x_2(t), x_3(t), x_4(t))$ of (1) is bounded on $R_+$ if and only if $x_3(0)=-\int_0^\infty e^{-s}f_3(s)\,ds$ and $x_4(0)=-\int_0^\infty e^{-s}f_4(s)\,ds$. A solution $x$ is bounded on $R_-$ if and only if $x_2(0)=\int_{-\infty}^0 e^{s}f_2(s)\,ds$ and $x_4(0)=\int_{-\infty}^0 e^{s}f_4(s)\,ds$.

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