A BOUNDED HILBERTIAN BASIS IN $C[0, 1]$

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Abstract. The existence of a bounded Hilbertian basis in $C[0, 1]$ is established.

1. Introduction. It is well known that an orthonormal sequence $\{x_n\}$ in a Hilbert space possesses the following properties:

(1) $\sum_{i=1}^\infty a_i x_i$ converges implies $\sum_{i=1}^\infty |a_i|^2 < \infty$, and

(2) $\sum_{i=1}^\infty |a_i|^2$ implies $\sum_{i=1}^\infty a_i x_i$ converges.

A basis $\{x_j\}$ in a Banach space satisfying (1) is called a Besselian basis and a basis $\{x_j\}$ in a Banach space satisfying (2) is called an Hilbertian basis.

A basis $\{x_n\}$ is bounded if $0 < \inf_n \|x_n\| \leq \sup_n \|x_n\| < \infty$. In [2], Pelczyński raised the question whether there exists in $C[0, 1]$ (in $L^1[0, 1]$) a bounded Besselian (resp. Hilbertian) basis, in particular, whether there exists in $C[0, 1]$ a bounded orthonormal basis. In [4], Olevskiĭ proved the non-existence of bounded orthonormal basis in $C[0, 1]$; in this paper we will show that there exists a bounded Hilbertian basis in $C[0, 1]$.

A basis $\{x_n\}$ of a Banach space is wc$_0$ (or semishrinking) if $x_n$ converges weakly to 0. Since every Hilbertian basis is a wc$_0$ basis, it is natural to look for a bounded Hilbertian basis in the class of bounded wc$_0$ bases. The referee has pointed out to us that Warren [3] has constructed a bounded wc$_0$ basis in $C[0, 1]$. Our basis is similar to his, but is a little simpler.

2. Main result. Let $a_0=0$, $a_1=1$ and for $j=2^n$, $a_{j+k}=(2k-1)/(2j)$ $(n=0, 1, \ldots; k=1, 2, \ldots, 2^n)$. Let $b_n=1/2^{n+1}$ $(n=1, 2, \ldots)$ and let $\{c_n\}$ be the subsequence of $\{a_n\}$ complementary to the subsequence $a_0, a_1, b_1, b_2, \ldots$ of $\{a_n\}$, i.e. $c_1=1/2$, $c_2=3/4$, $c_3=1/8$, $c_4=5/8$, $c_5=7/8$, $c_6=3/16$, \ldots. Next we define a rearrangement of $\{a_n\}$ as follows: $a_0, a_1, c_1$; $b_1, b_2, b_3, c_2; \ldots; b_{q(n)+1}, \ldots, b_{q(n)}, c_n; \ldots$, where $q(n)=1+2^2+\cdots+n^2-n(n=2, 3, \ldots)$. (Note. To each positive integer $n$ is associated a group of numbers consisting of $n^2-1$ $b_j$'s and one $c_i$.) To simplify matters, let us rename the above sequence as $d_0, d_1, d_2, \ldots$.

We now define a sequence in $C[0, 1]$ as follows (it is slightly different

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from the usual way of defining a generalized Schauder basis [1, p. 11]):

\[ x_0(t) = 1, \]
\[ x_1(t) = 1 - t, \]
\[ x_2(t) = 1, \quad t = d_2, \]
\[ = 0, \quad t \in \{d_0, d_1,d_3,...,d_{n-1}, g_n\}, \]
\[ = \text{linear for other } t, \]

and for \( n \geq 3 \)

\[ x_n(t) = 1, \quad t = d_n, \]
\[ = 0, \quad t \in \{d_0, d_1,d_3,...,d_{n-1}, g_n\}, \]
\[ = \text{linear for other } t, \]

where \( g_n \) denotes the first \( b_i \) in the sequence \( \{b_i\} \) not in \( \{d_0, d_1,d_3,...,d_n\} \).

Finally, we define another sequence in \( C[0, 1] \) as follows: \( y_i = x_i \quad (i = 0, 1, 2) \), and for \( n \geq 2 \) and \( p(n) = 1 + 2^2 + \cdots + (n-1)^2 + 1 \)

\[ y_{p(n)+1} = -x_{p(n)+1} - x_{p(n)+2} - \cdots - x_{p(n)+1} + n^{-2} x_{p(n)+1}, \]
\[ y_{p(n)+k} = x_{p(n)+k-1} + n^{-2} x_{p(n)+1} \quad (k = 2, \cdots, n^2). \]

We note in particular that for \( n \geq 3, 1 \leq k \leq n^2 \), the support of \( y_{p(n)+k} \) is contained in the union of two closed disjoint intervals \( I_{n,1} \) and \( I_{n,2} \), where \( I_{n,1} = [b_q(n)+1, b_q(n-1)] \) and \( I_{n,2} = [b_q(n-1)-1, 1] \). (Note: \( b_q(n-1)-1 \leq b_{n-1} < c_n < 1 \).)

**Theorem.** The sequence \( \{y_n\} \) is a bounded Hilbertian basis of \( C[0, 1] \).

**Proof.** The direct method used in showing the generalized Schauder basis of \( C[0, 1] \) is a basis [1, p. 11] can be used with slight modification to show \( \{x_n\} \) is a basis. Since \( x_i = y_i \quad (i = 0, 1, 2) \), \( x_{p(n)+1} = y_{p(n)+1} + \cdots + y_{p(n)+1} \), and \( x_{p(n)+k} = y_{p(n)+k+1} + n^{-2} y_{p(n)+1} \quad (k = 1, \cdots, n^2-1) \), \( [x_i] = [y_i] \quad (i = 0, 1, 2) \) and \( [x_{p(n)+1}, \cdots, x_{p(n)+1}] = [y_{p(n)+1}, \cdots, y_{p(n)+1}] \). Hence it suffices [1, p. 64] to show that there is a constant \( C \) independent of \( n \) such that for any sequence of real numbers \( h_1, h_2, \cdots, \)

\[
\left\| \sum_{i=1}^{k} h_i y_{p(n)+i} \right\| \leq C \left\| \sum_{i=1}^{k+1} h_i y_{p(n)+i} \right\| (n = 1, 2, \cdots; k = 1, 2, \cdots, n^2).
\]

The following argument is similar to an argument used in [3]; we present it here for completeness. Let \( g(k) = 0 \) if \( k = n^2 \) and \( = 1 \) if \( 1 \leq k < n^2 \), then

\[
\left\| \sum_{i=1}^{k} h_i y_{p(n)+i} \right\| = \text{Max}\{\mid h_1 + \cdots + h_k \mid n^2, \mid h_1 - h_2 \mid, \cdots, \mid h_1 - h_k \mid, g(k) \mid h_1 \mid\} \leq 2 \text{Max}\{\mid h_i \mid \mid 1 \leq i \leq n^2 \}.
\]
Now let $e_1 = (h_1 + \cdots + h_N)/N$, and $e_k = h_k - h_1$, $k = 2, \cdots, n^2 = N$; then
$h_1 = e_1 - (e_1 + e_2 + \cdots + e_n)/N$, $h_k = e_k + h_1$ ($k = 2, \cdots, n^2 = N$), and
\[
\max\{|h_i| : 1 \leq i \leq n^2\} \leq 3 \max\{|e_i| : 1 \leq i \leq n^2\}.
\]
Hence for $N = n^2$
\[
\left\| \sum_{i=1}^{N} h_i y_p(n) + i \right\| = \max\{|h_1 + \cdots + h_N|/N, |h_1 - h_2|, \cdots, |h_1 - h_N|\}
\]
\[
= \max\{|e_1| : 1 \leq i \leq N\}
\]
\[
\geq \frac{1}{2} \max\{|h_i| : 1 \leq i \leq N\}
\]
\[
= \frac{1}{6} \left\| \sum_{i=1}^{k} h_i y_p(n) + i \right\|.
\]
Hence $C$ can be chosen to be 6.

To show that $\{y_n\}$ is a bounded Hilbertian basis, we may disregard a finite number of $y_n$'s. Therefore, for sake of symmetry, we will consider only $y_{p(n)+k}$ for $n \geq 3$, $1 \leq k \leq n^2$. Let $I_{n,1}$ and $I_{n,2}$ be the two disjoint closed intervals mentioned above. It is clear that
\[
\sup\{|y_{p(n)+k}(t)| : t \in I_{n,1}\} = 1,
\]
and
\[
\sup\{|y_{p(n)+k}(t)| : t \in I_{n,2}\} = n^{-2} \quad (n \geq 3, 1 \leq k \leq n^2).
\]
For $n \geq 3$, $1 \leq k \leq n^2$, let
\[
u_p(n)+k(t) = y_{p(n)+k}(t), \quad t \in I_{n,1},
\]
\[= 0 \quad \text{otherwise},
\]
\[
u_p(n)+k(t) = y_{p(n)+k}(t), \quad t \in I_{n,2},
\]
\[= 0 \quad \text{otherwise}.
\]
One can readily verify that $u_{p(n)+1} = -(x_{p(n)+1} + \cdots + x_{p(n+1)-1})$, $u_{p(n)+k} = x_{p(n)+k-1}$ ($k = 2, \cdots, n^2$), and $v_{p(n)+k} = n^{-2} x_{p(n)+1}$ ($k = 1, 2, \cdots, n^2$). Thus $u_{p(n)+k}$, $v_{p(n)+k}$ are in $C[0,1]$, $u_{p(n)+k} + v_{p(n)+k} = y_{p(n)+k}$, $\|u_{p(n)+k}\| = 1$, $\|v_{p(n)+k}\| = n^{-2}$, $u_{p(n)+1} \leq 0$ and $u_{p(n)+k} \geq 0$ ($2 \leq k \leq n^2$). And because of the way the $x_i$'s are defined, we also have
\[
\left\| \sum_{n=3}^{N} u_{p(n)+1} \right\| = \left\| \sum_{n=3}^{N} \sum_{j=1}^{n^2-1} x_{p(n)+j} \right\| = 1.
(N \geq 3), and similarly,

\left\| \sum_{n=3}^{N-1} \sum_{k=2}^{2^k} u_{p(n)+k} + \sum_{k=2}^{2^k} u_{p(N)+k} \right\| = 1

(N \geq 3, 2 \leq K \leq N^2).

Let f \in C[0, 1]^*. Then \( f(y_n) = f(u_n) + f(v_n) \) (n \geq 7), and

\begin{align*}
\sum_{n=7}^{\infty} |f(y_n)|^2 & \leq 2 \left\{ \sum_{n=7}^{\infty} |f(u_n)|^2 + \sum_{n=7}^{\infty} |f(v_n)|^2 \right\}.
\end{align*}

Hence it is sufficient to prove the two series on the right converge. Now \( |f(v_n)| \leq \|f\| \|v_n\| = \|f\|/k^2 \) for some \( k \geq 3 \). But because of the way the \( y_n \)'s are constructed, there are exactly \( k^2 \) \( v_n \)'s with \( \|v_n\| = k^{-2} \). Hence

\begin{align*}
\sum_{n=7}^{\infty} |f(v_n)|^2 & \leq \|f\|^2 \sum_{k=3}^{\infty} k^2 \left( \frac{1}{k^2} \right)^2 < \infty.
\end{align*}

To show the other series converges, we note that there exists some function \( h \) of bounded variation on \([0, 1]\) such that \( f(u_n) = \int_0^1 u_n \, dh \). Let \( h_1 \) and \( h_2 \) be the positive and negative variations of \( h \) respectively. For any integer \( M = p(N) + K \geq 7 \) (\( N \geq 3, 1 \leq K \leq N^2 \)),

\begin{align*}
\sum_{n=7}^{M} |f(u_n)| & = \sum_{n=3}^{N-1} \sum_{k=1}^{n^2} |f(u_{p(n)+k})| + \sum_{k=1}^{K} |f(u_{p(N)+k})| \\
& = \sum_{n=3}^{N} |f(u_{p(n)+1})| + \sum_{n=3}^{N-1} \sum_{k=2}^{n^2} |f(u_{p(n)+k})| + \sum_{k=2}^{K} |f(u_{p(N)+k})| \\
& \leq 2 \sum_{i=1}^{2} \sum_{n=3}^{N} \left| \int_0^1 u_{p(n)+1} \, dh_i \right| \\
& \quad + 2 \sum_{i=1}^{2} \sum_{n=3}^{N} \sum_{k=2}^{n^2} \left| \int_0^1 u_{p(n)+k} \, dh_i \right| + \sum_{k=2}^{K} \left| \int_0^1 u_{p(N)+k} \, dh_i \right| \\
& = 2 \int_0^1 \left( \sum_{n=3}^{N} u_{p(n)+1} \right) \, dh_i \\
& \quad + 2 \sum_{i=1}^{2} \int_0^1 \left( \sum_{n=3}^{N} \sum_{k=2}^{n^2} u_{p(n)+k} + \sum_{k=2}^{K} u_{p(N)+k} \right) \, dh_i \\
& \leq 2(V(h_1) + V(h_2))
\end{align*}

where \( V(h_i) \) (i=1, 2) are the total variations of the monotone increasing functions \( h_i \). This obviously implies the convergence of the series.

Finally, the boundedness is clear. This completes the proof.
REFERENCES


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