AN INVARIANT OF CONFORMAL MAPPINGS

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Abstract. A result of W. Blaschke on conformal invariants of a surface is generalized.

1. Introduction. A conformal mapping on euclidean \( m \)-space \( E^m \) can be decomposed into a product of similarity transformations and inversions \( \{ \pi_j \} \) (Haantjes [2]). Let \( M \) be a surface in \( E^m \). If the center of inversion \( \pi_i \) of the conformal mapping does not lie on the surface \( M \) for all \( \pi_i \), then the conformal mapping is called a conformal mapping of \( E^m \) with respect to \( M \). A quantity on \( M \) is called a conformal invariant if it is invariant under conformal mappings of \( E^m \) with respect to \( M \).

The main purpose of this note is to prove the following

**Theorem.** Let \( M \) be a surface in \( E^m \) with Gauss curvature \( K \), mean curvature \( H \) and volume element \( dV \). Then \( (H^2 - K) dV \) is a conformal invariant.

If the codimension is one, this theorem was given by Blaschke [1].

2. Proof of the Theorem. It is obvious that the quantity \( (H^2 - K) dV \) is invariant under similarity transformations (motions and homothetics on \( E^m \)). Hence, it suffices to prove the Theorem for inversions. Let \( \pi \) be an inversion on \( E^m \) such that the center of \( \pi \) does not lie on the surface \( M \). We choose the origin at the center of the inversion \( \pi \). Let \( x \) and \( \bar{x} \) be the position vectors of the origin surface \( M \) and the inverse surface \( \bar{M} \) respectively, and let \( c \) be the radius of inversion \( \pi \). Then we have

\[
\bar{x} = \left( \frac{c^2}{r^2} \right) x, \quad r^2 = x \cdot x.
\]

From this we find that

\[
d\bar{x} = \left( \frac{c^2}{r^2} \right) dx - \left( 2c^2/r^3 \right) (dr)x,
\]

\[
d\bar{x} \cdot d\bar{x} = \left( \frac{c^4}{r^4} \right) dx \cdot dx.
\]
Hence the volume element $d\bar{V}$ of $\bar{M}$ is given by
\begin{equation}
46 \quad d\bar{V} = \left(\frac{c^4}{r^4}\right) dV.
\end{equation}

Let $e_3, \ldots, e_{m-2}$ be any $m-2$ mutually orthogonal unit normal local vector fields on $M$. Then
\begin{equation}
56 \quad \bar{e}_\alpha = \frac{2(x \cdot e_\alpha)}{r^2}x - e_\alpha, \quad \alpha = 3, \ldots, m-2,
\end{equation}
are $m-2$ mutually orthogonal unit normal vector fields on $\bar{M}$. From (2) and (5), we obtain
\begin{equation}
62 \quad dx \cdot d\bar{e}_\alpha = \left(\frac{2c^2(x \cdot e_\alpha)}{r^2}\right) dx \cdot dx - \left(\frac{c^2}{r^2}\right) dx \cdot dx.
\end{equation}

Combining (3) and (6), we find that, for any unit vector $e$ of $M$ in $E^m$, the principal curvatures $k_i(e)$, $i = 1, 2$, of $M$ with respect to $e$ satisfy the following
\begin{equation}
70 \quad \bar{k}_i(e) = -(r^4/c^4)k_i(e) - \left(\frac{2r^2}{c^2}\right)(x \cdot e), \quad i = 1, 2,
\end{equation}
where $\bar{k}_i(e)$ are the corresponding principal curvatures on $\bar{M}$ and $\bar{e} = \frac{2(x \cdot e)}{r^2}x - e$. Hence we obtain
\begin{equation}
80 \quad \left(\bar{k}_1(e) + \bar{k}_2(e)\right)^2 - 4\bar{k}_1(e)\bar{k}_2(e) = (r^4/c^4)((k_1(e) + k_2(e))^2 - 4k_1(e)k_2(e)).
\end{equation}

By taking averages of both sides of (8) over the spheres of unit normal vectors of $\bar{M}$ and $\bar{M}$ at the corresponding points, we obtain
\begin{equation}
90 \quad \bar{H}^2 - R = \left(\frac{r^4}{c^4}\right)(\bar{H}^2 - K),
\end{equation}
where $\bar{H}$ and $R$ are the mean curvature and the Gauss curvature of $\bar{M}$. Hence, from (4) and (9), we obtain the Theorem.

**Remark 1.** If $M$ is an orientable closed surface in $E^m$, then, by combining the Theorem and the well-known Gauss-Bonnet formula, we see that the integral $\int_M H^2 dV$ is a global conformal invariant. If the codimension is one, this invariant was observed by White [3].

**References**


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