BOUNDED HOLOMORPHIC FUNCTIONS IN SIEGEL DOMAINS

SU-SHING CHEN

Abstract. A Siegel domain $D$ of the second kind (not necessarily affine homogeneous) is shown to be complete with respect to the Carathéodory distance. Thus $D$ is convex with respect to the bounded holomorphic functions, hence is a domain of holomorphy. A Phragmén-Lindelöf theorem for $D$ is also given. That is, if a holomorphic function $f$ in $D$ is continuous in $\bar{D}$, bounded on the distinguished boundary $S$ of $D$ and not of exponential growth, then $f$ is bounded in $D$.

1. Introduction. The purpose of this paper is to prove two theorems concerning bounded holomorphic functions in Siegel domains of the second kind.

Theorem 1. A Siegel domain $D$ of the second kind (not necessarily affine homogeneous) is complete with respect to the Carathéodory distance on $D$.

Corollary 1. A Siegel domain $D$ of the second kind (not necessarily affine homogeneous) is convex with respect to the bounded holomorphic functions, hence is a domain of holomorphy.

Corollary 2. A bounded homogeneous domain in $\mathbb{C}^n$ is convex with respect to the bounded holomorphic functions, hence is a domain of holomorphy.

Theorem 2 (Phragmén-Lindelöf). Let $f(z, u)$ be a holomorphic function in a Siegel domain $D$ of the second kind and continuous in $\bar{D}$. Suppose that $|f(z, u)| \leq M$ on the distinguished boundary $S$ of $D$ and $f(z, u)$ is not of exponential growth. Then $|f(z, u)| \leq M$ in $\bar{D}$.

2. Siegel domains of the second kind. Let $\Omega$ be a regular cone in $\mathbb{R}^n$, i.e. a nonempty convex open set such that $0 \neq x \in \Omega$ and $\lambda > 0$ imply $\lambda x \in \Omega$, $-x \notin \Omega'$. $\Omega'$ denotes the dual cone, i.e. the set of all real linear...
functions $\alpha$ on $\mathbb{R}^n$ such that $\langle x, \alpha \rangle > 0$ for all $0 \neq x \in \Omega$. Let $\Phi: \mathbb{C}^m \times \mathbb{C}^m \rightarrow \mathbb{C}^n$ be a hermitian bilinear form with respect to $\mathbb{R}^n$ such that $\Phi(u, u) \in \Omega$ for all $u \in \mathbb{C}^m$ and let $D \subset \mathbb{C}^{n+m}$ be the domain

$$D = \{(z, u) \in \mathbb{C}^{n+m} \mid \text{Im} \ z = \Phi(u, u) \in \Omega\}.$$  

$D$ is called a Siegel domain of the second kind. If $m=0$, then $D$ is called a Siegel domain of the first kind or a radial tubular domain over $\Omega$. The distinguished boundary $S$ is the set

$$\{(z, u) \in \mathbb{C}^{n+m} \mid \text{Im} \ z = \Phi(u, u)\}.$$  

It is known that if $F$ is a bounded continuous function in $D$ and holomorphic in $D$, then $\sup_D |f(z, u)| = \sup_S |f(z, u)|$. A theorem of Gindikin, Pyatetskii-Shapiro and Vinberg [6] says that every bounded homogeneous domain in $\mathbb{C}^n$ is biholomorphic to an affine homogeneous Siegel domain of the second kind.

3. **Bounded holomorphic convexity of Siegel domains.** Let $d_M$ and $c_M$ denote the Kobayashi and Carathéodory pseudodistances respectively on a complex manifold $M$. (For definitions, see [4].) A complex manifold is said to be convex with respect to the bounded holomorphic functions (convex with respect to $B(M)$) if

$$K_B = \{x \in M \mid |f(x)| \leq \|f\|_K, \text{ for all } f \in B(M)\}$$

($B(M)$ the algebra of bounded holomorphic functions on $M$) is compact provided $K$ is a compact subset of $M$. A theorem of S. Kobayashi [4] says that if $M$ is complete with respect to the Carathéodory distance, then $M$ is convex with respect to $B(M)$. Here we shall prove that a Siegel domain $D$ of the second kind is complete with respect to the Carathéodory distance on $D$. The proof is quite trivially implied by Kobayashi's book [4]. However this fact is still worthwhile to be pointed out. For instance, the well-known theorem that a bounded homogeneous domain is a domain of holomorphy is a corollary of this fact. Another consequence is that a radial tubular domain is a domain of holomorphy [1]. Moreover, convexity with respect to $B(M)$ is much stronger than holomorphy. The domain $\{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1| < |z_2| < 1\}$ is a domain of holomorphy but is not convex with respect to $B(M)$. In [3], D. S. Kim has shown that convexity with respect to $B(M)$ implies bounded holomorphy. A domain of bounded holomorphy is a maximal domain for which every bounded holomorphic function has a bounded analytic continuation. The punctured disk $0 < |z| < 1$ is a domain of holomorphy but is not a domain of bounded holomorphy. The domain $H \times \mathbb{C}^{n-1} \subset \mathbb{C}^n$, where $H = \{z \in \mathbb{C} \mid \text{Re} \ z > 0\}$, is a domain of bounded holomorphy but is not convex with respect to $B(M)$. 
Proof of Theorem 1. According to [4, p. 64], a Siegel domain of the second kind can be written as the intersection of (possibly uncountably many) domains each of which is biholomorphic to a product of balls. A product of balls is known to be complete hyperbolic with respect to the Kobayashi distance. Since for a bounded symmetric domain the Kobayashi and Carathéodory distances coincide, a product of balls is complete with respect to the Carathéodory distance. Let $M$ and $M_i$ $(i \in I)$ be complex submanifolds of a complex manifold $N$ such that $M = \bigcap_{i \in I} M_i$. If each $M_i$ is complete with respect to its Carathéodory distance, so is $M$. (See [4].) Consequently, a Siegel domain of the second kind is complete with respect to the Carathéodory distance.

4. A Phragmén-Lindelöf theorem. Let $f$ be a holomorphic function in a domain $D$ of the complex plane $\mathbb{C}^1$ between two straight lines making an angle at the origin and continuous in $\overline{D}$. Suppose that $|f(z)| \leq M$ on the lines. The Phragmén-Lindelöf theorem says that either $|f(z)| \leq M$ in $\overline{D}$ or $f$ is of exponential growth [8]. We say that a function $f$ is not of exponential growth or

$$|f(z, u)| = o(\exp(|z_1|^\gamma + \cdots + |z_n|^\gamma + \langle \Phi(u, u), \alpha \rangle)),$$

if

$$|f(z, u)| \exp(-(|z_1|^\gamma + \cdots + |z_n|^\gamma + \langle \Phi(u, u), \alpha \rangle)) \to 0$$

whenever $\sum_k |z_k| + \sum_j |u_j| \to \infty$, for $(z, u) \in D$, a fixed $\alpha \in \Omega'$ and a fixed number $\gamma$ $(0 < \gamma < 1)$.

Proof of Theorem 2. Without loss of generality, we may assume that our domain $D$ is $\{(z, u)|\text{Re } z - \Phi(u, u) \in \Omega\}$ by a rotation. Consider the function $F(z, u) = \exp(-\varepsilon(z_1 + \cdots + z_n)^2) \exp(-\varepsilon(z, \alpha) f(z, u)$, where $\varepsilon$ is a positive real number. Then $F$ is holomorphic in $D$, because the first factor is holomorphic in a larger domain containing $D$. This larger domain (see [2], [6]) is the product of

$$\text{Re } z_1 - (|u_1|^2 + \cdots + |u_{m_1}|^2) > 0,$$

$$\cdots$$

$$\text{Re } z_n - (|u_{m_{n-1}}|^2 + \cdots + |u_{m_n}|^2) > 0.$$

Let $z_k = r_k e^{i\theta_k}$, then $-(\pi/2) \leq \theta_k \leq (\pi/2)$ in $\overline{D}$. On $S$, we have

$$|F(z, u)| = \exp(-\varepsilon(r_1^2 \cos \gamma \theta_1 + \cdots + r_n^2 \cos \gamma \theta_n))$$

$$\cdot \exp(-\varepsilon(\text{Re } z, \alpha) \cdot |f(z, u)|)$$

$$= \exp(-\varepsilon(r_1^2 \cos \gamma \theta_1 + \cdots + r_n^2 \cos \gamma \theta_n))$$

$$\cdot \exp(-\varepsilon(\Phi(u, u), \alpha) \cdot |f(z, u)|$$

$$\leq |f(z, u)| \leq M.$$
Moreover, for \((z, u) \in D\),
\[
|F(z, u)| \leq \exp\left(-\varepsilon (r_1 \cos \gamma \theta_1 + \cdots + r_n \cos \gamma \theta_n)\right)
\cdot \exp\left(-\varepsilon \langle \Phi(u, u), \alpha \rangle\right) \cdot |f(z, u)| \to 0,
\]
whenever \(\sum_k |z_k| + \sum_j |u_j| \to \infty\) by the assumption. Consequently \(F\) is bounded and continuous in \(\bar{D}\) and holomorphic in \(D\). Since \(|F(z, u)| \leq M\) on \(S\), \(|F(z, u)| \leq M\) in \(\bar{D}\). Let \(\varepsilon \to 0\). We obtain finally \(|f(z, u)| \leq M\) for all \((z, u) \in \bar{D}\).

References


Department of Mathematics, University of Florida, Gainesville, Florida 32601