ANOSOV FLOWS AND EXPANSIVENESS

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Abstract. We prove that an Anosov diffeomorphism of a compact manifold is expansive. We also show that a continuous flow on an infinite compact metric space cannot be expansive. We define a corresponding expansive concept for flows, that of an unstable flow (the word unstable is used here in a Lyapunov sense). We then prove that Anosov flows of compact manifolds are unstable.

1. Introduction. Let $X$ be a metric space with metric $d$ and $G$ a topological group. A transformation group $(X, G, \pi)$ is called expansive if there is an $\varepsilon > 0$ (called an expansive constant) such that for any pair of distinct points $x, y$ in $X$ there is a $g \in G$ such that $d(xg, yg) > \varepsilon$. Here, as is customary, $xg$ is used in place of $\pi(x, g)$.

If $G$ is the integers then the transformation group is generated by a homeomorphism $f: X \to X$. In this case $f$ is called an expansive homeomorphism. There are numerous examples of expansive homeomorphisms on compact spaces [2]. On the other hand if $G$ is the reals $\mathbb{R}$ and $X$ is an infinite compact space then $(X, \mathbb{R}, \pi)$ cannot be expansive (Proposition 3.1). When $G = \mathbb{R}$, the transformation group will be called a flow. For flows, we will use $\gamma(x)$ to denote the orbit of the point $x$.

Definition 1.1. A flow $\pi: X \times \mathbb{R} \to X$ is called unstable if there is an $\varepsilon > 0$ such that for any $x, y$ in $X$ with $y \notin \gamma(x)$ there is a $t$ (possibly negative) for which $d(xt, yt) > \varepsilon$. Thus, if $\pi$ is unstable, no orbit can be both positively and negatively Lyapunov stable (here we assume $X$ is self-dense; $X$ will be a manifold for the remainder of the paper). In §2 we prove that an Anosov diffeomorphism on a compact manifold is expansive. This is indicated in [3, Proposition 8.7] but a proof has not been published. In §3 we prove that an Anosov flow on a compact manifold is unstable.

2. Anosov diffeomorphisms. A diffeomorphism $f$ of a compact Riemannian manifold $M$ is called an Anosov diffeomorphism if there is a splitting of the tangent bundle $TM = E^u + E^s$, invariant under the derivative $Df$,
and positive numbers \(a, b,\) and \(\lambda\) such that:

\[
\| Df^n \cdot \xi^u \| \geq ae^{\lambda n} \| \xi^u \|, \quad |Df^{-n} \cdot \xi^u| \leq be^{-\lambda n} \| \xi^u \|, \\
\| Df^n \cdot \xi^s \| \leq be^{-\lambda n} \| \xi^s \|, \quad |Df^{-n} \cdot \xi^s| \geq ae^{\lambda n} \| \xi^s \|,
\]

for every \(\xi = \xi^u + \xi^s\) in \(TM\).

We will assume that

\[
\| Df \cdot \xi^u \| \geq 7 \| \xi^u \|, \quad |Df^{-1} \cdot \xi^u| \leq \frac{7}{2} \| \xi^u \|, \\
\| Df \cdot \xi^s \| \leq \frac{3}{2} \| \xi^s \|, \quad |Df^{-1} \cdot \xi^s| \geq 7 \| \xi^s \|,
\]

for each \(\xi\) in \(TM\). These inequalities hold for some iterate of \(f\), and \(f\) is expansive if that iterate of \(f\) is expansive.

This last set of inequalities implies that, given any tangent vector \(\xi\) in \(TM\), \(\| Df \cdot \xi^u \| \geq 3 \| \xi^u \|\) if \(\| \xi^u \| \geq \| \xi^s \|\) and \(\| Df^{-1} \cdot \xi^u \| \geq 3 \| \xi^u \|\) if \(\| \xi^s \| \geq \| \xi^u \|\). For instance when \(\| \xi^u \| \geq \| \xi^s \|\) we have

\[
\| Df \cdot \xi \| = \| Df \cdot \xi^u + Df \cdot \xi^s \| \geq \| Df \cdot \xi^u \| - \| Df \cdot \xi^s \| \\
\geq 7 \| \xi^u \| - \frac{7}{2} \| \xi^s \| \geq 6 \| \xi^u \| \geq 3 \| \xi^s \|.
\]

Let us denote the zero cross-section of \(TM\) by \(0_*\). For \(r > 0\) and \(p\) in \(M\), \(B(r)_p\) is the open ball of radius \(r\) about \(0\) in \(TM_p\). We let \(B(r)\) denote the open tube of radius \(r\) about \(0_0\) in \(TM\), \(B(r) = \bigcup B(r)_p\) (\(p \in M\)).

Let \(\pi : TM \to M\) be the projection and \(\exp\) the exponential map. The map \(\Psi : \xi \to (\pi \xi, \exp \xi)\) restricts to a diffeomorphism of some open neighborhood of \(0_*\) in \(TM\) onto an open neighborhood of the diagonal in \(M \times M\). In order to simplify later measurements we lift \(f\) and \(f^{-1}\) to \(TM\) via \(\Psi\), defining \(F = \Psi^{-1} \cdot (f \times f) \cdot \Psi\), \(G = \Psi^{-1} \cdot (f^{-1} \times f^{-1}) \cdot \Psi\). Then if \(r\) is taken sufficiently small, \(F\) and \(G\) restrict to diffeomorphisms of \(B(r)\) onto open neighborhoods \(U\) and \(V\) of \(0_*\) in \(TM\). We assume further that \(d(\pi \xi, \exp \xi) = \| \xi \|\) for any \(\xi\) in \(B(r) \cup U \cup V\).

Let \(F_p\) and \(G_p\) denote the restrictions of \(F\) and \(G\) to the fibre \(B(r)_p\). Then \(F_p\) and \(G_p\) are maps of Euclidean spaces. We have the relations:

\[
DF_p(0) \cdot \xi = Df \cdot \xi, \quad DG_p(0) \cdot \xi = Df^{-1} \cdot \xi
\]

for every \(p\) in \(M\) and \(\xi\) in \(TM_p\). These equations are a consequence of the chain rule and the fact that \(D(\exp_p)(0) : TM_p \to TM_p\) is the identity map.

**Lemma 2.1.** Let \(f : M \to M\) be an Anosov diffeomorphism of the compact Riemannian manifold \(M\). There is a positive number \(\varepsilon\) such that either \(d(f(p), f(q)) \geq 2d(p, q)\) or \(d(f^{-1}(p), f^{-1}(q)) \geq 2d(p, q)\) whenever \(d(p, q) < \varepsilon\).

**Proof.** The map \(\xi \to \| DF \xi(\xi) - DF \xi(0) \|\) of \(B(r)\) into \(R\) is continuous and by compactness we can choose a number \(\varepsilon_F\) less than or equal to \(r\) such that \(\| DF \xi(\xi) - DF \xi(0) \| \leq 1\) for any \(\xi\) in \(B(\varepsilon_F)\) (the norm of the linear transformation \(A = DF \xi(\xi) - DF \xi(0)\) of \(TM_\xi\) is the usual sup norm, \(\| A \| = \sup \{ \| A \cdot \eta \| : \| \eta \| \leq 1\}\)). Likewise choose \(\varepsilon_G\) less than or equal
to \( r \) such that \( \| DG_{\xi}(\xi) - DG_{\xi}(0)\| \leq 1 \) when \( \xi \) is in \( B(\epsilon_G) \). Then setting 
\[ \epsilon = \min\{\epsilon_F, \epsilon_G\}, \]
we have the estimates [1, I, §4, Corollary 2],
\[ \| F_p(\xi) - Df \cdot \xi \| \leq \| \xi \|, \quad \| G_p(\xi) - Df^{-1} \cdot \xi \| \leq \| \xi \|, \]
for every \( p \) in \( M \) and \( \xi \) in \( B(\epsilon) \).

Now suppose \( p \) and \( q \) are given with \( d(p, q) < \epsilon \). Take \( \xi \) in \( B(\epsilon) \) satisfying \( \exp f = a \). If \( \| \xi \| \leq \| \xi \| \),
\[ d(f(p), f(q)) = \| F_p(\xi) \| \geq \| Df \cdot \xi \| - \| \xi \| \]
\[ \geq 3 \| \xi \| - \| \xi \| = 2 \| \xi \| = 2d(p, q). \]

On the other hand if \( \| \xi \| \geq \| \xi \| \) we have \( d(f^{-1}(p), f^{-1}(q)) \geq 2d(p, q) \) by an analogous computation.

Applying Lemma 2.1 repeatedly we have:

**Theorem 2.2.** An Anosov diffeomorphism of a compact manifold \( M \) is expansive (the \( \epsilon \) of Lemma 2.1 is an expansive constant).

3. **Anosov flows.** In this section \( M \) is again a compact Riemannian manifold and the notations are the same as in §2.

**Proposition 3.1.** A flow on an infinite compact space cannot be expansive.

**Proof.** Let \( X \) be infinite and compact and \( \phi : X \times I \to X \) a flow. Let \( \epsilon \) be greater than 0, \( I = [0, 1] \), and \( \psi = \phi |_{X \times I} \). Then if \( \rho \) is the product metric on \( X \times I \), there is a \( \delta > 0 \) such that \( \rho((y, t_1), (z, t_2)) < \delta \) implies \( d(y t_1, z t_2) < \epsilon \). If \( \phi \) is to be expansive there can be only finitely many rest points. Let \( x \) be a nonrest point and choose a sequence of positive numbers \( \{t_n\} \) decreasing monotonically to 0 so that \( x \neq xt_n \) for any \( n \). If \( \epsilon \) is an expansive constant for each \( n \) there is an \( s_n \) such that \( d(xs_n, (xt_n)s_n) > \epsilon \). However for \( n \) large enough \( \rho((xs_n, 0), (xs_n, t_n)) < \delta \) and hence \( d(xs_n, (xt_n)s_n) = d(xs_n, (xt_n)s_n) < \epsilon \).

A differentiable flow \( \phi : M \times R \to M \) induced by a vector field \( \mathcal{V} \) is an Anosov flow if

(i) \( \phi \) has no rest points,

(ii) there is an invariant decomposition \( TM = E^s + E^u + E^0 \) satisfying the following: there exist positive constants \( a, b, \lambda \) such that for any \( \xi = \xi^s + \xi^u + \xi^0 \) in \( TM \),
\[ \| D\phi_t \cdot \xi^u \| \geq a e^{\lambda t} \| \xi^u \| \quad \text{for all} \ t \geq 0, \]
\[ \| D\phi_t \cdot \xi^s \| \leq b e^{-\lambda t} \| \xi^s \| \quad \text{for all} \ t \geq 0, \]
\[ \| D\phi_t \cdot \xi^u \| \leq b e^{-\lambda t} \| \xi^u \| \quad \text{for all} \ t \leq 0, \]
\[ \| D\phi_t \cdot \xi^s \| \geq a e^{\lambda t} \| \xi^s \| \quad \text{for all} \ t \leq 0. \]
(iii) $E^0$ has dimension one and is tangent to the flow.

Note we can always choose a positive number $T$ so that

$$\|D\phi_T \cdot \xi^u\| > 7 \|\xi^u\|, \quad \|D\phi_{-T} \cdot \xi^u\| < \frac{1}{7} \|\xi^u\|, \quad (3.2)$$

$$\|D\phi_T \cdot \xi^s\| < \frac{1}{7} \|\xi^s\|, \quad \|D\phi_{-T} \cdot \xi^s\| > 7 \|\xi^s\|.$$ 

There is a local codimension one submanifold $N_p$ at $p$ whose tangent space at $p$ is $E^s_p + E^u_p$. $N_p$ is transverse to the trajectories of the flow and is the image under exp$_p$ of a neighborhood of $0$ in $E^s_p + E^u_p$.

**Lemma 3.3.** Let $\phi: M \times \mathbb{R} \to M$ be an Anosov flow. There exist $T$ and $\varepsilon$ greater than $0$ such that if $p$ is in $M$ and $q$ is in $N$ and $d(p, q) < \varepsilon$ then either $d(\phi_T(p), \phi_T(q)) \geq 2d(p, q)$ or $d(\phi_{-T}(p), \phi_{-T}(q)) \geq 2d(p, q)$.

**Proof.** Let $T$ be chosen so that the inequalities, 3.2, are satisfied. Then the proof is essentially the same as that of Lemma 2.1 with $\phi_T$ replacing $f$ and $\phi_{-T}$ replacing $f^{-1}$. In particular, the $\varepsilon$ of this lemma, and of the following theorem, is small enough so that the exponential map is defined on the $\varepsilon$-tube about the zero section of the tangent bundle of $M$.

**Theorem 3.4.** An Anosov flow on a compact manifold $M$ is unstable.

**Proof.** Let $\phi: M \times \mathbb{R} \to M$ be an Anosov flow and let $\varepsilon$ and $T$ be as in the preceding lemma. Choose $\alpha > 0$ such that if $d(p, q) < \alpha$ then $d(\phi_T(p), \phi_T(q)) < \varepsilon/2$ and $d(\phi_{-T}(p), \phi_{-T}(q)) < \varepsilon/2$. Now choose $\eta > 0$ such that if $d(p, q) < \eta$ then there is a $q_p$ in $\gamma(q)$ with $q_p$ in $N_p$ and $d(q, q_p) < \alpha$. The existence of $\eta$ is guaranteed by the fact that trajectories are transverse to $N_p$. Let $\delta = \min(\eta, \varepsilon/2)$. Suppose $p, q$ are in $M$, $q \notin \gamma(p)$ and $d(p, q) < \delta$. By the lemma either $d(\phi_T(p), \phi_T(q_p))$ or $d(\phi_{-T}(p), \phi_{-T}(q_p))$ is greater than $2d(p, q_p)$. Also $d(\phi_{kT}(q), \phi_{kT}(q_p)) < \varepsilon/2$. Thus for some integer $k$ we must have either $d(\phi_{kT}(p), \phi_{kT}(q)) > \eta$ and we cannot guarantee the existence of $q_p$ or else $d(\phi_{kT}(p), \phi_{kT}(q)) > \varepsilon/2$ and the exponential map is no longer defined. Thus $\delta$ is an instability constant for $\phi$ and the proof is completed.

**References**


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