LOCAL AUTOMORPHISMS ARE DIFFERENTIAL OPERATORS ON SOME BANACH SPACES

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Abstract. If $E$ belongs to a certain category of Banach spaces (the $B^\infty$-smooth spaces) which include Hilbert spaces and if $F$ is any normed space, it is proved that any local linear automorphism of $C^\infty(E, F)$ is a differential operator. This generalizes a result of J. Peetre when $E=R^n$ and $F=R$.

1. A result of J. Peetre ([2], [3]) is the following characterization of linear partial differential operators:

A linear map $T$ of $C^\infty(R^n, R)$ into $C^\infty(R^n, R)$ is a linear "partial differential operator" if and only if $T$ is local i.e. for each $f \in C^\infty(R^n, R)$, support($Tf$) $\subseteq$ support($f$).

It should be noted that by a linear partial differential operator $T$ is meant a collection $\{A_a \in C^\infty(R^n, R)\}$ such that the sets

$$G_a = \{x \in R^n \mid A_a(x) \neq 0\}$$

from a locally finite collection and such that $T(f)(x) = \sum A_a(x) D^a(f)(x)$ for each $x \in R^n$ and each $f \in C^\infty(R^n, R)$. Here $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index, $|\alpha| = \sum_{i=1}^n \alpha_i$ and

$$D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}.$$ 

In this paper we prove that at least for $E$ in a certain category of Banach spaces this theorem extends to local (linear) automorphisms on $C^\infty(E, F)$ where $C^\infty(E, F)$ now denotes the infinitely Fréchet differentiable $F$-valued functions on $E$ and $F$ is any normed linear space. Defining $L^k(E, F)$ to be the bounded symmetric $k$-multilinear maps from $E$ to $F$ we have $D^\alpha f(x) \in L^k(E, F)$ for each $x$.

A natural generalization of a finite dimensional differential operator to
an arbitrary Banach space is
\[ T(f)(x) = \sum_{i=0}^{\infty} \alpha_i(x)(D^i f(x)) \]
where \( \alpha_i \in C^\infty(E, L(L^2(E, F), F)) \) and the supports of the \( \alpha_i \) form a locally finite collection. Such maps are clearly local linear automorphisms on \( C^\infty(E, F) \).

As in Wells [6] we let
\[
B^k(E, F) = \left\{ f \mid f \in C^k(E, F), \sup_{x \neq y} \| D^k f(x) - D^k f(y) \| / \| x - y \| < \infty \right\}
\]
and
\[
B^\infty(E, F) = \bigcap_{k=0}^{\infty} B^k(E, F).
\]

An isomorphic invariant of a Banach space due to Bonic and Frampton [1] is \( C^p \) smoothness. \( E \) is \( C^p \) smooth, \( p = 0, 1, 2, \ldots, \infty \), if there is some \( \eta \in C^p(E, R) \) with \( \eta(0) \neq 0 \) and \( \eta(\{ x \mid \| x \| = 1 \}) = 0 \). Similarly, as in [6], \( E \) will be called \( B^p \) smooth, \( j = 0, 1, 2, \ldots, \infty \), if there is an \( \eta \in B^p(E, R) \) with \( \eta(0) \neq 0 \) and \( \eta(\{ x \mid \| x \| = 1 \}) = 0 \). \( B^\infty \) smooth spaces have been called uniformly \( C^\infty \) smooth in Quinn [4]. Finite dimensional spaces as well as \( L^p \) for \( p \) an even integer are \( B^\infty \) smooth. \( l^1 \) is not \( C^1 \) smooth. \( c_0 \) is \( C^\infty \) smooth but not \( B^1 \) smooth. Separable \( C^p \) or \( B^p \) smooth Banach spaces admit partitions of unity of class \( C^p \) or \( B^p \) respectively. In these cases \( C^p(E, F) \) or \( B^p(E, F) \) is dense in \( C^p(E, F) \) or \( B^p(E, F) \) respectively for any \( B \)-space \( F \). Refer to Bonic and Frampton [1], Wells ([5] and [6]) for more details.

2. Theorem. If \( E \) and \( F \) are Banach space, if \( E \) is \( B^\infty \) smooth and if \( T: C^\infty(E, F) \to C^\infty(E, F) \) is a local linear map, then \( T \) is a differential operator in the sense described above.

The proof will require three lemmas. Only the first will use the \( B^\infty \) smoothness of \( E \). We will use \( K_r(x) \) to denote the open ball of radius \( r \) centered at \( x \).

Lemma 1. Let \( x_0 \in E \). There is a neighborhood \( U_{x_0} \) of \( x_0 \) and an integer \( k \) with that property that if \( f, g \in C^\infty(E, F) \), \( y \in U_{x_0} \) and \( D^i f(y) = D^i g(y) \) for \( 0 \leq i \leq k \) then \( T(f)(y) = T(g)(y) \).

Proof. If this were not the case there would be an \( x_0 \in E \), a sequence \( x_n \) tending to \( x_0 \) and a sequence \( f_n \in C^\infty(E, F) \) with \( D^k f_n(x_n) = 0 \) for \( k \leq n \) and \( \| T(f_n)(x_n) \| = n \). By the \( B^\infty \) smoothness there exists an \( \eta \in B^\infty(E, R) \) with \( \eta(\text{cl}(K_{1/\sqrt{2}}(0))) = 1 \) and \( \eta(\{ x \mid \| x \| \geq 1 \}) = 0 \). Let \( N = \sup_x \| D^i \eta(x) \| \). For
each \( n \) there is an \( M_n \) and an \( r_n \) such that \( \| D^j f_n(x) \| \leq M_n (\| x - x_n \|)^{n+1-j} \) for \( x \in K_{r_n}(x_n) \) and \( 0 \leq j \leq n \). Whenever \( 1/a_n < r_n \) we have

\[
\sup_x \| D^i (f_n(x) \eta(a_n(x - x_n))) \| \\
\leq \sum_{i=0}^j \left( \frac{j}{i} \right) \sup_{x \in K_{1/a_n}(x_n)} \| D^i f_n(x) \| \cdot a_n \cdot \sup_x \| D^i (\eta(x)) \| \\
\leq \sum_{i=0}^j \left( \frac{j}{i} \right) M_n \cdot \left( \frac{1}{a_n} \right)^{n+1-i} \cdot a_n \cdot N_i = \left( \frac{1}{a_n} \right)^{n+1-j} M_n \sum_{i=0}^j \left( \frac{j}{i} \right) N_i.
\]

Thus we can choose a sequence \( a_n \) so that

(i) \( 1/a_n < r_n \),

(ii) \( K_{1/a_n}(x_n) \cap K_{1/a_n}(x_m) = \emptyset \) for \( n \neq m \),

(iii) \( \sup_{j \leq n, x \in E} \| D^j (f_n(x) \eta(a_n(x - x_n))) \| < \text{dist}(x_0, K_{1/a_n}(x_n)) \).

It follows that the function \( f(x) = \sum_{n=1}^\infty f_n(x) \eta(a_n(x - x_n)) \) belongs to \( C^\infty(E, F) \) and that \( f(x) \equiv f_n(x) \) for \( x \in K_{1/2a_n}(x_n) \). Consequently

\[ \| T(f)(x) \| = n \]

so that \( T(f) \) is not a continuous function at \( x_0 \). This is a contradiction.

Let \( E_k = F \oplus L^1_k(E, F) \oplus \cdots \oplus L^k(E, F) \). By Lemma 1 for each \( x \in U_{x_0} \) there is a linear map \( T_x : E_k \to F \) such that

\[ T(f)(x) = T_x(f(x), Df(x), \cdots, D^k f(x)). \]

In Lemmas 2 and 3, \( x_0 \) and \( U_{x_0} \) will be fixed.

**Lemma 2.** \( T_x \) is bounded except possibly at a set \( I_{x_0} \) of isolated points of \( U_{x_0} \).

**Proof.** If this were not the case there would exist a sequence \( \{x_n\} \) with \( \{x_n\} \subset U_{x_0} \) and a \( y \in U_{x_0} \) with \( y = \lim x_n \) and with \( T_{x_n} \) unbounded for each \( n \). Next we choose a collection \( \{\varphi_n\} \subset B^\infty(E, R) \) with support \( \varphi_n \cap \text{support } \varphi_m = \emptyset \) for \( n \neq m \), \( \text{dist}(x_0, \text{support } \varphi_n) > 0 \) for all \( n \), and \( \varphi_n(x) = 1 \) near \( x_n \). (We observe that the \( B^\infty \) smoothness of \( E \) is not needed to construct the \( \{\varphi_n\} \) since the \( x_n \)'s can be separated by a disjoint collection of weak neighborhoods each of which is the support of a \( B^\infty \) function equal to 1 near \( x_n \).) For each \( n \) choose \( g_n \in C^\infty(E, F) \) such that

\[ \sup_{j \leq n, x \in E} \| D^j (g_n(x) \varphi_n(x)) \| < \text{dist}(x_0, \text{support } \varphi_n) \]

and \( \| T_{x_n}(\{g_n(x_n), \cdots, D^k g_n(x_n)\})\| \geq n \). The function

\[ f(x) = \sum_{n=0}^\infty g_n(x) \varphi_n(x) \]
belongs to $C^\infty(E, F)$ and $f(x) = g_n(x)$ near $x_n$. Consequently $\|T(f)(x_n)\| \geq n$ which is impossible in view of the continuity of the function $T(f)$.

Thus $T_x$ induces a map $T^0 : U_{x_0} \setminus I_{x_0} \to L(E_k, F)$ such that $T^0(x) = T_x$. Hence $T^0(x)(f(x), \ldots, D^p f(x)) = T(f)(x)$ for $x \in U_{x_0} \setminus I_{x_0}$.

**Lemma 3.** For each $p = 0, 1, 2, \ldots$ and each $y_0 \in U_{x_0}$ there is a neighborhood $U_{y_0}$ of $y_0$ such that $T^0|U_{y_0} \setminus I_{x_0} \in B^p(U_{y_0} \setminus I_{x_0}, L(E_k, F))$.

**Proof.** In Wells [6] it is shown that

$$B^p(E, F) = \left\{ f \mid f \in C^0(E, F), \sup_{x, h \neq 0} \| \Delta^p_h f(x) \|/\| h \|^p+1 < \infty \right\}$$

where $\Delta^p_h f(x) = f(x + h) - f(x)$. Suppose the lemma were false. Then for some $p$ and some $y_0 \in U_{x_0}$ and for every neighborhood $N$ of $y_0$ contained in $x_0$, the supremum of $\| \Delta^p_h f(x) \|/\| h \|^p+1$ over all $x, h \neq 0$ with $x, x + h, \ldots, x + (p + 1)h$ contained in $N \setminus I_{x_0}$ would be infinite. This would imply the existence of sequences $\{x_n\}, \{h_n\}$ with $x_n \to y_0, h_n \to 0$,

$$\{x_n, x_n + h_n, \ldots, x_n + (p + 1)h_n\} \in U_{x_0} \setminus I_{x_0}$$

and

$$\| \Delta^p_h T^0(x_0) \|/\| h \|^p+1 \geq 4^n.$$

Choose $A_n \in E_k$ with $\| A_n \|_{E_k} \leq 3^{-n}$ and

$$\| \Delta^p_h T^0(x_0)(A_n) \| \geq \frac{3}{4} \| \Delta^p_h T^0(x_0) \| \cdot 3^{-n}.$$

Since for any $t, s$ in a normed linear space $\sup\{\| t + \sigma s \| \mid \sigma = \pm 1 \| \geq \| s \|$, we may inductively choose $\sigma_n = \pm 1, n = 1, 2, 3, \ldots$, so that

$$\| \Delta^p_h T^0(x_0) \left( \sum_{j=1}^n \sigma_j A_j \right) \| \geq \frac{3}{4} \| \Delta^p_h T^0(x_0) \| \cdot 3^{-n}.$$

For each $n$ let $g_n$ be the $k$ polynomial such that $A_n = \{g_n(x_n), \ldots, D^k g_n(x_n)\}$ and $f(x)$ be the $k$ polynomial $\sum_{i=1}^\infty \sigma_j g_j(x)$. Then

$$\| \Delta^p_h T(f)(x_n) \| = \| \Delta^p_h T^0(x_n)(f(x_n), \ldots, D^p f(x_n)) \| \geq \| \Delta^p_h T^0(x_n) \left( \sum_{j=1}^n \sigma_j A_j \right) \| - \| \Delta^p_h T^0(x_n) \left( \sum_{j=n+1}^\infty \sigma_j A_j \right) \| \geq \| \Delta^p_h T^0(x_n) \| \cdot \left( 3^{-n} - \sum_{j=n+1}^\infty 3^{-j} \right) = \frac{1}{4} 3^{-n} \cdot \| \Delta^p_h T^0(x_n) \| \geq \frac{1}{4} (4/3)^n \cdot \| h \|^p+1.$$

But this is a contradiction since, for every $p$, $T(f)$ is $B^p$ in some neighborhood of $y_0$. Q.E.D.
We are now in a position to prove the theorem. First observe that the choice of \( p = 0 \) in Lemma 3 implies that the exceptional set \( I_{x_0} \) of Lemma 2 is void. Hence \( T^0 \) is defined on all of \( U_{x_0} \) and by Lemma 3 is locally \( B^p \) for any \( p \) so that \( T^0 \in C^\infty(U_{x_0}, L(E_k, F)) \). Consequently there exist \( \alpha^0_n \in C^\infty(U_{x_0}, L(L^n(E, F), F)), n = 0, 1, \cdots, k \), such that

\[
T(f)(x) = \sum_{n=0}^{k} \alpha^0_n(D^n f(x))
\]

for all \( x \in U_{x_0} \). Suppose that \( T(f)(x) = \sum_{n=0}^{k'} \alpha_n'(D^n f(x)) \) for \( x \in U_{x_1} \) with \( \alpha_n' \in C^\infty(U_{x_1}, L(L^n(E, F), F)) \). Without loss of generality we may assume \( k = k' \). If \( x \in U_{x_0} \cap U_{x_1} \) and \( A \in L^n(E, F) \) for \( n \leq k \), then for \( g(x) = (1/n!) A(x, x, \cdots, x) \) we find \( \alpha_n'(A) = T(g)(x) = \alpha_n'(A) \). Hence on \( U_{x_0} \cap U_{x_1} \), \( \alpha_n' \) and \( \alpha_n' \) agree, so that we may define maps

\[
\alpha_n \in C^\infty(E, L(L^n(E, F), F)), \quad n = 0, 1, \cdots,
\]

such that \( (Tf)(x) = \sum_{n=0}^{\infty} \alpha_n(x)(D^n f(x)) \) for \( x \in E \) and the \( \{\alpha_n\} \) have locally finite supports. Consequently \( T \) is a differential operator.

**Bibliography**


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