

## A PRIME-DIVISOR FUNCTION

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**ABSTRACT.** This note studies the asymptotic mean values over arithmetical progressions, the general distribution of values, and the maximum order of magnitude, of a certain natural prime-divisor function of positive integers.

Consider the multiplicative arithmetical function  $\beta$  defined by  $\beta(1)=1$  and  $\beta(n)=\alpha_1\alpha_2\cdots\alpha_r$  if  $n=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}$  ( $p_i$  prime,  $\alpha_i>0$ ). Kendall and Rankin [2, p. 199] pointed out that this function has the finite *mean value*

$$\lim_{N\rightarrow\infty} \frac{1}{N} \sum_{n=1}^N \beta(n) = \frac{\zeta(2)\zeta(3)}{\zeta(6)} = 1.943 \cdots$$

Strangely, perhaps, there appears to be virtually no other information available about this natural arithmetical function. (See note added in proof.) This note makes a more detailed study of its asymptotic properties.

### 1. Average values and distribution.

**THEOREM 1.** *Let  $r$  and  $q$  denote relatively prime positive integers. Then.*

$$\begin{aligned} \sum_{n \leq x; n \equiv r \pmod{q}} \beta(n) = & \frac{1}{q} \left\{ \frac{L(2, \chi_0)L(3, \chi_0)}{L(6, \chi_0)} x + \frac{L(\frac{1}{2}, \chi_0)L(\frac{3}{2}, \chi_0)}{L(3, \chi_0)} x^{1/2} \right. \\ & + \frac{L(\frac{1}{3}, \chi_0)L(\frac{2}{3}, \chi_0)}{L(2, \chi_0)} x^{1/3} + \frac{\bar{\chi}_1(r)L(\frac{1}{2}, \chi_1)L(\frac{3}{2}, \chi_1)}{L(3, \chi_0)} x^{1/2} \\ & \left. + \sum_{\chi_2} \frac{\bar{\chi}_2(r)L(\frac{1}{3}, \chi_2)L(\frac{2}{3}, \chi_2)}{L(2, \chi_0)} x^{1/3} \right\} \\ & + O(x^{3/10} \log^{9/10} x \phi(q) q^{8/5}), \end{aligned}$$

where  $\chi_0$  denotes the principal character mod  $q$ , and the terms in  $\chi_1, \chi_2$  occur if and only if there exist characters  $\chi_1 \neq \chi_0, \chi_2 \neq \chi_0 \pmod{q}$  such that  $\chi_1^2 = \chi_0, \chi_2^3 = \chi_0$ .

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PROOF. Given any character  $\chi$  mod  $q$ , the Euler product formula for  $L(z, \chi)$  leads to the equation

$$\sum_{n=1}^{\infty} \chi(n)\beta(n)n^{-z} = \frac{L(z, \chi)L(2z, \chi^2)L(3z, \chi^3)}{L(6z, \chi^6)} \quad [\text{Re } z > 1].$$

Therefore, if  $L(z, \chi)L(2z, \chi^2)L(3z, \chi^3) = \sum_{n=1}^{\infty} c(n, \chi)n^{-z}$  [ $\text{Re } z > 1$ ], then

$$\sum_{n \leq x} \chi(n)\beta(n) = \sum_{km^6 \leq x} c(k, \chi)\chi^6(m)\mu(m) = \sum_{m \leq x^{1/6}} \chi^6(m)\mu(m) \sum_{k \leq x/m^6} c(k, \chi).$$

Now, by equations (68–71) of Richert [7], we have:

- (i) 
$$\sum_{n \leq x} c(n, \chi_0) = \frac{\phi(q)}{q} \{xL(2, \chi_0)L(3, \chi_0) + x^{1/2}L(\frac{1}{2}, \chi_0)L(\frac{3}{2}, \chi_0) + x^{1/3}L(\frac{1}{3}, \chi_0)L(\frac{2}{3}, \chi_0)\} + O(x^{3/10} \log^{9/10} x \phi(q)q^{8/5});$$
- (ii) 
$$\sum_{n \leq x} c(n, \chi_1) = \frac{\phi(q)}{q} x^{1/2}L(\frac{1}{2}, \chi_1)L(\frac{3}{2}, \chi_1) + O(x^{3/10} \log^{9/10} x \phi(q)q^{4/3})$$
 if  $\chi_1^2 = \chi_0$  but  $\chi_1 \neq \chi_0$ ;
- (iii) 
$$\sum_{n \leq x} c(n, \chi_2) = \frac{\phi(q)}{q} x^{1/3}L(\frac{1}{3}, \chi_2)L(\frac{2}{3}, \chi_2) + O(x^{3/10} \log^{9/10} x \phi(q)q^{3/5})$$
 if  $\chi_2^3 = \chi_0$  but  $\chi_2 \neq \chi_0$ ;
- (iv) 
$$\sum_{n \leq x} c(n, \chi) = O(x^{3/10} \log^{9/10} x q^{8/5})$$
 for all other  $\chi$ .

It follows for example that

$$\begin{aligned} \sum_{n \leq x} \chi_1(n)\beta(n) &= \sum_{m \leq x^{1/6}} \chi_1^6(m)\mu(m) \left\{ \frac{\phi(q)x^{1/2}}{qm^3} L(\frac{1}{2}, \chi_1)L(\frac{3}{2}, \chi_1) \right. \\ &\quad \left. + O((x/m^6)^{3/10} \log^{9/10}(x/m^6)\phi(q)q^{4/3}) \right\} \\ &= \frac{\phi(q)}{q} x^{1/2}L(\frac{1}{2}, \chi_1)L(\frac{3}{2}, \chi_1) \left[ \frac{1}{L(3, \chi_0)} + O(x^{-1/3}) \right] \\ &\quad + O(x^{3/10} \log^{9/10} x \phi(q)q^{4/3}) \\ &= \frac{\phi(q)}{q} x^{1/2} \frac{L(\frac{1}{2}, \chi_1)L(\frac{3}{2}, \chi_1)}{L(3, \chi_0)} + O(x^{3/10} \log^{9/10} x \phi(q)q^{4/3}). \end{aligned}$$

Similarly one can determine estimates for  $\sum_{n \leq x} \chi(n)\beta(n)$  in the other cases. The theorem then follows from the equation

$$\sum_{n \leq x; n \equiv r \pmod q} \beta(n) = \sum_{n \leq x} \beta(n) \frac{1}{\phi(q)} \sum_{\chi \pmod q} \chi(n)\bar{\chi}(r).$$

COROLLARY.

$$\sum_{n \leq x} \beta(n) = \frac{\zeta(2)\zeta(3)}{\zeta(6)} x + \frac{\zeta(\frac{1}{2})\zeta(\frac{3}{2})}{\zeta(3)} x^{1/2} + \frac{\zeta(\frac{1}{3})\zeta(\frac{2}{3})}{\zeta(2)} x^{1/3} + O(x^{3/10} \log^{9/10} x).$$

REMARK. By using a theorem of Schmidt [5] concerning  $\sum_{n \leq x} c(n)$  where  $\sum c(n)n^{-z} = \zeta(z)\zeta(2z)\zeta(3z)$  [ $\text{Re } z > 1$ ], one can sharpen the error term in this corollary to  $O(x^{7/27} \log^2 x)$ . For a more general, though less sharp, version of the corollary, see the author [3]. (See note added in proof.)

Next, by applying a theorem of Schoenberg [6, p. 319], one obtains

THEOREM 2. *The function  $\beta$  possesses an asymptotic distribution function*

$$F(x) = \lim_{N \rightarrow \infty} \frac{1}{N} \text{card}\{n \leq N : \beta(n) \leq x\}.$$

This function is discrete, and the characteristic function of  $F(e^x)$  is

$$\int_{-\infty}^{\infty} e^{itx} dF(e^x) = \prod_{\text{primes } p} (1 - p^{-1}) \left\{ 1 + \sum_{r=1}^{\infty} p^{-r} e^{it \log p^r} \right\}.$$

REMARK. As part of a different discussion, J. Ridley and the author [4] have shown that, for each  $k = 1, 2, \dots$ , the function  $\beta$  has a finite  $k$ th moment

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N [\beta(n)]^k = \prod_{\text{primes } p} \left\{ 1 + \sum_{r=2}^{\infty} [r^k - (r-1)^k] p^{-r} \right\}.$$

It may also be mentioned that a slight modification of a technique of Kendall and Rankin [2, p. 204] (who are concerned with the total number  $a(n)$  of nonisomorphic abelian groups of order  $n$ ) leads to an explicit formula for the frequency

$$F_m = \lim_{N \rightarrow \infty} \frac{1}{N} \text{card}\{n \leq N : \beta(n) = m\}.$$

Since this formula is relatively involved, we do not give it in detail, but we note that by combining certain of the frequencies  $P_m$  calculated for  $a(n)$  in [2, p. 205] one obtains:

$$\begin{aligned} F_1 &= P_1 = 6/\pi^2 = 0.6079 \dots, & F_2 &= P_2 = 0.2008 \dots, \\ F_3 &= P_3 = 0.0742 \dots, & F_4 &= P_4 + P_5 = 0.0542 \dots, \\ F_5 &= P_7 = 0.0147 \dots, & F_6 &= P_6 + P_{11} = 0.0215 \dots. \end{aligned}$$

Thus  $F_1 + \dots + F_6 = 0.9733 \dots$ , which emphasizes how closely the values of  $\beta$  cluster about its mean value 1.943  $\dots$ . In fact, after a computer

check on the “empirical” frequencies of  $\beta$  over the range  $1 \leq n \leq 10,000$ , J. Ridley has very kindly provided the following figures for the actual frequencies in this range:  $F'_1=0.6083$ ,  $F'_2=0.2008$ ,  $F'_3=0.0744$ ,  $F'_4=0.0541$ ,  $F'_5=0.0151$ ,  $F'_6=0.0216$ . Here  $F'_1 + \dots + F'_6 = 0.9743$ .

**2. Maximum order of magnitude.**

**THEOREM 3.** *Given any  $\varepsilon > 0$  there exists an integer  $n_0(\varepsilon)$  such that*

$$\beta(n) < 3^{(1/3)(1+\varepsilon)\log n / \log \log n} \text{ for all } n \geq n_0(\varepsilon),$$

while

$$\beta(n) > 3^{(1/3)(1-\varepsilon)\log n / \log \log n} \text{ for infinitely many } n.$$

**PROOF.** The argument is parallel to one whereby Hardy and Wright [1] proved an analogous theorem for the divisor function  $d(n)$ . Firstly, one notes by induction that  $a \leq 3^{a/3}$  for  $a=1, 2, \dots$ . Hence, for  $p \geq 3^{1/3\delta}$  ( $\delta > 0$ ) and  $a=1, 2, \dots$ ,

$$\frac{a}{p^{a\delta}} \leq \frac{a}{3^{a/3}} \leq 1.$$

If  $n = p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}$  ( $p_i$  prime,  $a_i > 0$ ), it follows that

$$\frac{\beta(n)}{n^\delta} = \prod_1^r \frac{a_i}{p_i^{a_i \delta}} \leq \prod_{\text{primes } p \leq 3^{1/3\delta}} (\delta \log 2)^{-1} < \exp\left(\frac{3^{1/3\delta}}{\delta \log 2}\right).$$

If  $\delta = (1 + \frac{1}{2}\varepsilon)\log 3/3 \log \log n$  ( $\varepsilon > 0$ ), then

$$\frac{3^{1/3\delta}}{\delta \log 2} = \frac{(\log n)^{1/(1+\varepsilon/2)} \log \log n}{(1 + \frac{1}{2}\varepsilon)\log 2 \cdot \log 3^{1/3}} < \frac{\varepsilon \log 3^{1/3} \cdot \log n}{2 \log \log n}$$

for  $n$  sufficiently large. Hence the upper inequality follows.

For the lower inequality, let  $N = (p_1 p_2 \dots p_r)^3$  where  $p_1 < \dots < p_r$  denote the first  $r$  primes. Then

$$\log \beta(N) = r \log 3 = \pi(p_r) \log 3 \geq \frac{\log(p_1 p_2 \dots p_r) \cdot \log 3}{\log p_r},$$

in a similar way to the situation in [1, p. 263]. Hence, in the same way, there is a constant  $C$  such that

$$\log \beta(N) > \frac{\log N \cdot \log 3^{1/3}}{\log \log N + C} > \frac{(1 - \varepsilon)\log 3^{1/3} \cdot \log N}{\log \log N}$$

for  $N$  sufficiently large, i.e. for  $r$  sufficiently large.

**THEOREM 4.** *Given any  $\varepsilon > 0$ ,  $\beta(n) < 3^{(1/3)(1+\varepsilon)\log \log n}$  for “almost all”  $n$ , i.e. all  $n$  outside some set of asymptotic density zero.*

PROOF. The inequality  $a \leq 3^{a/3}$  ( $a=1, 2, \dots$ ) implies that  $\beta(n) \leq 3^{\Omega(n)/3}$  where  $\Omega(n)$  is the sum of the exponents of the prime divisors of  $n$ . The theorem then follows from Theorem 431 of [1], which states that  $\Omega(n)$  has "normal order"  $\log \log n$ .

REMARK. One cannot expect a similar lower inequality, since for example  $\beta$  takes the value 1 on all square-free integers, and these have positive density  $6/\pi^2$ .

NOTE ADDED IN PROOF. In a recent paper, *The number of square-full divisors of an integer*, Proc. Amer. Math. Soc. **34** (1972), 79–80, D. Suryanarayana and R. Sita Rama Chandra Rao established the above corollary to Theorem 1, with an error estimate slightly weaker than that obtainable with the aid of Schmidt's theorem [5].

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