PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 40, Number 2, October 1973

ON CONTRACTIVE MAPPINGS

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ABSTRACT. The Cauchy condition for convergence of a contractive iteration in a complete metric space is replaced by an equivalent functional condition. Many generalizations of the Banach-Neumann contractive mapping principle follow.

Introduction. Let X be a complete metric space and let $f: X \to X$ be a contractive mapping or contraction, i.e.,

$$d(f(x), f(y)) < d(x, y) \quad \text{for all } x, y \text{ in } X.$$

Let x_0 be a chosen point in X and set $x_n = f(x_{n-1})$ for $n > 0$. Criteria for the sequence of iterates $\{x_n\}$ to be Cauchy are then of interest, for if it is Cauchy one can easily prove it converges to a unique fixed point x_∞ of the map f on X.

Many papers have presented such criteria, especially since the important paper of Rakotch [**1**]. In this note we first give in §1 a necessary and sufficient condition that a sequence of iterates be convergent. This condition is used in §2 to provide a criterion for the convergence of the iteration from an arbitrary initial point in X. The condition is then restated in a more familiar functional form, and some corollaries are then drawn in §3.

1. **Main theorems.** The following results, while stated for complete metric spaces, are readily extended to separated complete uniform spaces, as in [**2**, p. 197].

We shall use the following notation. For any pair of sequences x_n and y_n in X with $x_n \neq y_n$, we write

$$d_n = d(x_n, y_n) \quad \text{and} \quad \Delta_n = d(f(x_n), f(y_n))/d_n.$$

We then have the following theorem.

THEOREM 1.1. *Let X be a complete metric space. Let $f: X \to X$ with $d(f(x), f(y)) < d(x, y)$, for all x, y in X. Let $x_0 \in X$ and set $x_n = f(x_{n-1})$ for $n > 0$. Then $x_n \to x_\infty$ in X, with x_∞ a unique fixed point of f, iff for any*

Received by the editors January 2, 1973.
AMS (MOS) subject classifications (1970). Primary 54H25; Secondary 47H10.
Key words and phrases. Contractive mapping, fixed points, convergence of iterations.

two subsequences x_{h_n} *and* x_{k_n} *with* $x_{h_n} \neq x_{k_n}$, *we have that*

$$\Delta_n \to 1 \quad \textit{only if} \quad d_n \to 0.$$

PROOF. First assume that $x_n \to x_\infty$ in X and let x_{h_n} and x_{k_n} be any two subsequences. Then clearly $d_n = d(x_{h_n}, x_{k_n}) \to 0$ and so the condition is satisfied.

Next assume the condition is satisfied for a given initial point x_0 in X. Then $d_n = d(x_n, x_{n+1})$ is a decreasing sequence of nonnegative numbers and so approaches some $\varepsilon \geqq 0$. Assume $\varepsilon > 0$. Then letting $h_n = n$ and $k_n = n+1$, we have $d_n \to \varepsilon > 0$ while $\Delta_n \to 1$. So the condition is violated. Thus $d(x_n, x_{n+1}) \to 0$.

Now assume the given sequence of iterates $\{x_n\}$ is not Cauchy. Then there exists some $\varepsilon > 0$ such that every tail $\{x_n\}_{n \geqq N}$ of the sequence has diameter $D_N = \sup_{n,m \geqq N} d(x_n, x_m) > \varepsilon$. Given this ε, we will construct a pair of subsequences violating the condition.

For any $n > 0$, let N_n be so large that $d(x_m, x_{m+1}) < 1/n$ for all $m \geqq N_n$, as is possible since $d(x_m, x_{m+1}) \to 0$. Let $h_n \geqq N_n$ be the lowest integer such that for some $k_n > h_n$, $d(x_{h_n}, x_{k_n}) > \varepsilon$. Such pairs exist by the above diameter condition. Next choose k_n to be the least such integer above h_n. Then either $k_n - 1 = h_n$ or else $d(x_{h_n}, x_{k_n - 1}) \leqq \varepsilon$. In either case we have $\varepsilon \leqq d_n = d(x_{h_n}, x_{k_n}) < \varepsilon + 1/n$.

Moreover, using the triangular inequality on the contraction, we have

$$1 \geqq \Delta_n = \frac{d(f(x_{h_n}), f(x_{k_n}))}{d_n} \geqq \frac{d_n - 2/n}{d_n}.$$

So $\Delta_n \to 1$ while $d_n \to \varepsilon > 0$, again violating the condition.

So $\{x_n\}$ must be a Cauchy sequence in X and as X is complete, we have $x_n \to x_\infty$ for some x_∞ in X. Then by the usual arguments, x_∞ is a unique fixed point of f and the proof is complete.

The above proof is essentially that of Theorem 6.1 in [**2**, p. 197] with the added observation that on a particular iteration, the condition is necessary. A close inspection of the proof shows that the following slight improvement was really proven.

COROLLARY 1.2. *Let X be a complete metric space. Let $f: X \to X$ with $d(f(x), f(y)) < d(x, y)$ for all x, y in X. Let $x_0 \in X$ and set $x_n = f(x_{n-1})$ for $n > 0$. Then $x_n \to x_\infty$ in X, with x_∞ a unique fixed point of f, iff for any two subsequences x_{h_n} and x_{k_n} with $x_{h_n} \neq x_{k_n}$, we have that $\Delta_n \to 1$, with d_n decreasing, only if $d_n \to 0$.*

PROOF. Same as Theorem 1.1.

We now wish to convert this sequential condition to the more customary functional form. Following Rakotch [1], we define a class of test functions more general than his.

DEFINITION. S is the class of functions $\alpha: \boldsymbol{R}^+ \to [0, 1)$ with

(i) $\boldsymbol{R}^+ = \{t \in \boldsymbol{R} \mid t > 0\}$,

(ii) $\alpha(t_n) \to 1$ implies $t_n \to 0$.

Notes. (1) We do not assume that α is continuous in any sense.

(2) We only require that if α gets near one, it does so only near zero.

(3) Using Corollary 1.2, we could replace property (ii) with (ii′) $\alpha(t_n) \to 1$ with t_n decreasing implies $t_n \to 0$.

THEOREM 1.3. *Let $f: X \to X$ be a contraction on a complete metric space. Let $x_0 \in X$ and set $x_n = f(x_{n-1})$ for $n > 0$. Then $x_n \to x_\infty$, where x_∞ is a unique fixed point of f in X, iff there exists an α in S such that for all n, m,*

$$d(f(x_n), f(x_m)) \leqq \alpha(d(x_n, x_m)) \cdot d(x_n, x_m).$$

PROOF. We need only show that the existence of such an α in S is equivalent to the sequential condition of Theorem 1.1. First assume such an α exists. Let x_{h_n} and x_{k_n} be subsequences with $x_{h_n} \neq x_{k_n}$. Assume that $\Delta_n \to 1$. Then it follows from the above inequality that $\alpha(d(x_{h_n}, x_{k_n})) \to 1$. But then since $\alpha \in S$, we have $d(x_{h_n}, x_{k_n}) \to 0$.

Next assume that the sequential condition holds. Define $\alpha: \boldsymbol{R}^+ \to \boldsymbol{R}$ as follows:

$$\alpha(t) = \sup\left\{\frac{d(f(x_n), f(x_m))}{d(x_n, x_m)} \,\middle|\, d(x_n, x_m) \geqq t\right\}.$$

Since f is a contraction, the quotients are all below 1 and so α is defined for all $t > 0$ and $\alpha \leqq 1$. Now assume that $\alpha(t_n) \to 1$ for $t_n \in \boldsymbol{R}^+$. We may further assume without loss of generality that $1 - 1/n < \alpha(t_n) \leqq 1$. We must show $t_n \to 0$. But $\alpha(t_n)$ is the above least upper bound. So there is for each $n > 0$ a pair x_{h_n}, x_{k_n} in $\{x_n\}$ with

$$d(x_{h_n}, x_{k_n}) \geqq t_n$$

and

$$1 - \frac{1}{n} < \frac{d(f(x_{h_n}), f(x_{k_n}))}{d(x_{h_n}, x_{k_n})} \leqq \alpha(t_n).$$

So the sequence $\Delta_n \to 1$. But then by the sequential condition of Theorem 1.1, $d(x_{h_n}, x_{k_n}) \to 0$. So $t_n \to 0$, as was to be shown. This completes the theorem.

2. **A general convergence criterion.** We may now apply the results of §1 to obtain a criterion for convergence of the iteration from an arbitrary

starting point. Unfortunately, some generality is lost and the universal condition, while still sufficient, is no longer necessary.

THEOREM 2.1. *Let $f: X \to X$ be a contraction of a complete metric space satisfying*

$$d(f(x), f(y)) \leqq \alpha(d(x, y)) \cdot d(x, y)$$

where $\alpha \in S$. Then for any choice of initial point x_0, the iteration $x_n = f(x_{n-1})$, $n > 0$, converges to the unique fixed point x_∞ of f in X.

PROOF. Simply apply Theorem 1.3 to any initial point x_0.

It should be noted that Theorem 2.1 is essentially the same as Corollary 1 of [**2**, p. 197]. The improvement afforded by the above presentation is that the essential character of the condition on each orbit is brought out in §1.

3. **Some corollaries.** The usefulness of the contraction mapping principle has prompted many generalizations of the original result, which is Theorem 2.1 with $\alpha \in S$ being simply a constant function. A few of these are listed below, all of them following easily from Theorem 2.1 (which, as pointed out in §2, is really little more than Corollary 1 of [**2**, p. 197]).

COROLLARY 3.1 (RAKOTCH [**1**, p. 463]). *If $f: X \to X$ is a contraction of a complete metric space satisfying $d(f(x), f(y)) \leqq \alpha(d(x, y)) \cdot d(x, y)$ where $\alpha: \boldsymbol{R}^+ \to [0, 1)$ and is monotone decreasing, then for any choice of x_0 in X, the iteration $x_n = f(x_{n-1})$, $n > 0$, converges to a unique fixed point x_∞ of f in X.*

PROOF. Such an α is clearly in the class S.

COROLLARY 3.2. *If $f: X \to X$ is a contraction of a complete metric space satisfying $d(f(x), f(y)) \leqq \alpha(d(x, y)) \cdot d(x, y)$ where $\alpha: \boldsymbol{R}^+ \to [0, 1)$ and is monotone increasing, then for any choice of x_0 in X, the iteration $x_n = f(x_{n-1})$, $n > 0$, converges to a unique fixed point x_∞ of f in X.*

PROOF. As in Corollary 3.1.

COROLLARY 3.3 (BOYD-WONG [**4**, p. 331]). *If $f: X \to X$ is a contraction of a complete metric space satisfying $d(f(x), f(y)) \leqq \alpha(d(x, y)) \cdot d(x, y)$ where $\alpha: \boldsymbol{R}^+ \to [0, 1)$ and is continuous, then for any choice of x_0 in X, the iteration $x_n = f(x_{n-1})$, $n > 0$, converges to a unique fixed point x_∞ of f in X.*

PROOF. As in Corollary 3.1.

REMARK 1. The last corollary is derived from Theorem 1 of [**3**, p. 459]. This latter theorem also seems related to Theorem 2.1 and is a corollary,

if we were to use the slightly stronger form provided by the Note (3) on the class of functions S in §1.

REMARK 2. A paper investigating the relations among these various corollaries will shortly be submitted to the Journal for Mathematical Analysis and Applications.

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