CRITERIA FOR COMPACTNESS AND FOR DISCRETENESS OF LOCALLY COMPACT AMENABLE GROUPS

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ABSTRACT. Let $G$ be a locally compact group $P(G)=\{0 \leq \phi \in L_1(G); \int \phi(x) \, dx = 1\}$ and $(l_\phi f)(x) = \phi f(ax)$ for all $a, x \in G$ and $f \in L^\infty(G)$. 0 $\leq \Psi \in L^\infty(G)^*$, $\Psi(1)=1$ is said to be a [topological] left invariant mean (LIM) if $\Psi(ab)=\Psi(f)$ $[\Psi(\phi \ast f)=\Psi(f)]$ for all $a \in G$, $\phi \in P(G)$, $f \in L^\infty(G)$. The main result of this paper is the

Theorem. Let $G$ be a locally compact group, amenable as a discrete group. If $G$ contains an open $\sigma$-compact normal subgroup, then $\text{LIM}=\text{TLIM}$ if and only if $G$ is discrete. In particular if $G$ is an infinite compact amenable as discrete group then there exists some $\Psi \in \text{LIM}$ which is different from normalized Haar measure. A harmonic analysis type interpretation of this and related results are given at the end of this paper.\footnote{The main result of this paper has independently been obtained by W. Rudin in a recent paper Invariant means on $L^\infty$. Studia Math. 44 (1972), 219–227, which was not in print when our paper was sent for publication. (See Addition at the end of present paper.)}

Introduction. It was known to Fred Greenleaf that if $T$ is the circle group then there are at least two different linear translation invariant functionals $\Psi \geq 0$ on $L^\infty(T)$ with $\Psi(1)=1$. One of them is certainly that given by the normalized Haar measure $\lambda$ on $T$.

It is easy to show and it is known that on any compact $G$, $\lambda$ is the unique $0 \leq \Psi \in L^\infty(G)^*$, $\Psi(1)=1$ which satisfies the stronger invariance property $\Psi(\phi \ast f)=\Psi(f)$ for all $f \in L^\infty(G)$, $\phi \in P(G)$ (i.e. $\lambda$ is the unique TLIM on $L^\infty(G)$). This is the case since $\phi \ast f \in C(G)$ for all $\phi \in P(G)$, $f \in L^\infty(G)$ and if $\Psi \in \text{TLIM}$ then $\Psi \in \text{LIM}$ [6, p. 25]. Thus $\Psi=\lambda$ at least on $C(G)$. But then for all $f \in L^\infty(G), \Psi(f)=\Psi(\phi \ast f)=\lambda(\phi \ast f)=\lambda(f)$.

It seemed to Greenleaf that for any compact infinite $G$, which is amenable as a discrete group, there exist at least two different LIM's on $L^\infty(G)$. Our main result in this paper implies the

Theorem. Let $G$ be a locally compact group which is abelian or $\sigma$-compact and amenable as a discrete group. Then $\text{LIM}=\text{TLIM}$ if and
only if $G$ is discrete. In particular on any compact infinite $G$ which is amenable as discrete there exists some $\Psi \in \text{LIM}$ different from the normalized Haar measure.

Let $H \{H_c\}$ be the linear span of $\{f-l_\alpha f; f \in L^\infty(G), \alpha \in G\}$ $[\{f-\phi \ast f; \phi \in P(G), f \in L^\infty(G)\}]$ and for $A \subset L^\infty(G)$ let $\overline{A} \{A^*\}$ denote the norm $[w^*]$ closure of $A$ in $L^\infty(G)$. In any locally compact group one has $\widehat{H} \subset \widehat{H}_c \subset \widehat{H}_c^* = \widehat{H}^* \subset L^\infty(G)$. Our last result (combined with some known facts) when restricted to $\sigma$-compact locally compact abelian groups runs as follows:

**Proposition.** (i) If $G$ is compact and infinite then $\overline{H} \subset \widehat{H}_c = \overline{H}_c^* = \overline{H}^* = \{f \in L^\infty(G); \lambda f = 0\}$.

(ii) If $G$ is not compact then $\overline{H} \subset \widehat{H}_c \subset \overline{H}_c^* = \overline{H}^* = L^\infty(G)$. Moreover $L^\infty(G)/\widehat{H}_c$ is a nonseparable Banach space and $\overline{H} = \widehat{H}_c$ iff $G$ is discrete.

We conjecture at the end that for any locally compact amenable group $G$, if $G$ is noncompact then $L^\infty(G)/\widehat{H}_c$ is a nonseparable Banach space and $\overline{H} = \widehat{H}_c$ iff $G$ is discrete.

Some more notations. Unless otherwise specified we assume the notations and definitions of Hewitt-Ross [7].

If $G$ is a locally compact group $\lambda$ will denote a fixed left Haar measure (with $\lambda(G) = 1$ if $G$ is compact), we write sometimes $\int \phi(x) \, dx$ instead of $\int \phi \, d\lambda$.

$\Psi \in L^\infty(G)^*$ is said to be [topologically] left invariant if $\Psi(l_\alpha f) = \Psi(f)$ $[\Psi(\phi \ast f) = \Psi(f)]$ for all $f \in L^\infty(G), \phi \in P(G), \alpha \in G$ (where $l_\alpha f(x) = \alpha f(x) = f(\alpha x)$). If $\Psi$ satisfies in addition $\Psi \geq 0$ and $\Psi(1) = 1$ then $\Psi$ is said to be a [topological] left invariant mean ([TLIM] LIM resp.). The set of all [TLIM] LIM is also denoted by [TLIM] LIM. Analogously we define [TRIM] RIM the sets of [topological] right invariant means.

We stress that LIM, TLIM are both included in $L^\infty(G)^*$. The locally compact group $G$ is said to be amenable if LIM $\neq \emptyset$ (or equivalently if TLIM $\neq \emptyset$ see [6]). $G$ is said to be amenable as discrete if $G_d$ (i.e. $G$ with the discrete topology) is amenable.

We write sometimes LIM$(G)$, TLIM$(G)$ to emphasize dependence on the group $G$. If $A \subset G$, $1_A$ denotes the function 1 on $A$ and zero otherwise. If $\Psi \in L^\infty(G)^*$, we write $\Psi(B)$ instead of $\Psi(1_B)$ for measurable $B \subset G$. 1 also stands for the constant one function on $G$.

**Proposition 1.** Let $G$ be any noncompact locally compact group and $\phi \in \text{TRIM}$. If $B$ is a measurable set and $\lambda(B) < \infty$ then $\phi(B) = 0$. 
Proof. Let $\phi_\alpha \in P(G)$ be such that $\phi_\alpha \to \phi$ in $w^*$ and let $\eta \in P(G)$ be such that $0 \leq \eta(x) \leq \varepsilon$ for all $x \in G$. Then
\[
|\phi_\alpha \ast \eta(x)| = \left| \int \phi_\alpha(y) \eta(y^{-1}x) \, d\lambda \right| \leq \varepsilon \int \phi_\alpha(y) \, d\lambda = \varepsilon.
\]
Furthermore if $f \in L^\infty(G)$ then
\[
(\phi_\alpha \ast \eta)(f) = \phi_\alpha(f \ast \eta) \to \phi(f \ast \eta) = \phi(f).
\]
(See Wong [10, p. 352].) Hence if $f \in L^\infty \cap L^1$ then $|(\phi_\alpha \ast \eta)(f)| \leq \int |f| \, d\lambda$ so $|\phi f| \leq (\int |f| \, d\lambda) \varepsilon$. Thus $\phi f = 0$.

We need the following, probably known, proposition for which we were unable to find a reference.

**Proposition 2.** Let $G$ be a $\sigma$-compact nondiscrete locally compact group. Then for any $\varepsilon > 0$ there exists an open dense set $B \subset G$ with $\lambda(B) < \varepsilon$.

Proof. It is enough to show the existence of a dense set $D \subset G$ with $\lambda(D) = 0$ and the regularity of $\lambda$ would imply that for some open $D \subset B$, $\lambda(B) < \varepsilon$.

If $G$ is separable then there is some countable dense $D \subset G$. Clearly $\lambda(D) = 0$.

Assume now that $G$ is arbitrary and $N \subset G$ a closed normal subgroup. Let $\theta : G \to G/N$ be the canonical map. If $D \subset G$ with $\theta D$ dense in $G/N$ then $DN$ is dense in $G$. In fact if $U \subset G$ is open with $U \cap DN = \emptyset$ then $UN \cap DN = \emptyset$ so $\theta^{-1}(U \cap \theta D) = UN \cap DN = \emptyset$ thus $\theta U \cap \theta D = \emptyset$ and $\theta U$ is open in $G/N$ which cannot be.

If $G$ is $\sigma$-compact nondiscrete let $U \subset G$ be an open neighborhood of the identity and let $G_0 = \bigcup_{n=1}^\infty U^n$. Then $G_0$ is open compactly generated and there are countably many left cosets of $G$ w.r.t. $G_0$. The left Haar measure of $G_0$ can be taken to be the restriction to $G_0$ of the left Haar measure $\lambda$ on $G$. It is enough hence to show that there is a dense null set $D \subset G_0$ i.e. we can and shall assume that $G$ is compactly generated nondiscrete. Let then $U_n$ be a sequence of identity neighborhoods in $G$ with $\lambda(U_n) \to 0$ and let $N \subset \bigcap_{n=1}^\infty U_n$ be a compact normal subgroup such that $G/N$ is metrizable separable (see [7, p. 71]). ($G/N$ is not discrete since $\lambda N = 0$ so $N$ is not open.) Let $D = \{d_i\}_{i=1}^\infty \subset G$ be such that its image in $G/N$ is dense. Then $DN \subset G$ is dense and $\lambda(DN) = 0$ since $D$ is countable.

We need in the sequel the following proposition (not in its full force) which is in part due to Følner [3] for discrete amenable groups.

**Proposition 3.** Let $G$ be a locally compact group which is amenable as a discrete group. For $f \in L^\infty(G)$ let $M(f) = \sup \{\phi(f) ; \phi \in LIM\}$. Then
for all \( f \in L^\infty(G) \)

\[
Mf = \inf_{\mathcal{A}} \text{ess sup} \left[ \frac{1}{n} \sum_{i=1}^{n} f(a_i, x) \right]
\]

the inf being taken over the set \( \mathcal{A} \) of all finite tuples \((a_1, \cdots, a_n)\) of elements of \( G \).

**Proof.** Let \( H \) be the linear span of \( \{ f - l_a f; \ a \in G, \ f \in L^\infty(G) \} \). It is known (and due to Følner [3, p. 6] for discrete \( G \)) that:

\[
M(f) = \inf_{\mathcal{A}} \text{ess sup} (f(x) + h(x))
\]

for all \( f \in L^\infty(G) \). (For an easy proof see [5, p. 401].)

Also if \( \phi \in \text{LIM} \) then \( \phi f = \phi(n^{-1} \sum_{i} l_{a_i} f) \) hence

\[
Mf \leq \inf_{\mathcal{A}} \text{ess sup} \frac{1}{n} \sum_{i} f(a_i, x).
\]

Let now \( \varepsilon > 0 \) and \( h_0 \in H \) be such that \( M(f) + \varepsilon > \text{ess sup}_{x}(f(x) + h_0(x)) \). So \( M(f) + \varepsilon \geq f(x) + h_0(x) \) locally a.e. and a fortiori \( M(f) + \varepsilon \geq n^{-1} \sum_{i} l_{a_i} (f(x) + h_0(x)) \) loc. a.e. for all \( a_1, \cdots, a_n \) in \( G \). We claim that a finite set \( \{ b_1, \cdots, b_k \} \subset G \) can be chosen such that \( |k^{-1} \sum l_{b_i} h_0(x)| < \varepsilon/2 \) loc. a.e. This would imply that \( M(f) + 3/2 \varepsilon \geq k^{-1} \sum l_{b_i} f(x) \) loc. a.e., i.e. that

\[
M(f) \geq \inf_{\mathcal{A}} \text{ess sup} \frac{1}{n} \sum_{i} l_{a_i} f(x)
\]

which would end this proof.

To prove this claim let \( h_0 = \sum_{i} [f_i - l_{c_i} f_i] \). For the finite set \( F = \{ c_1, \cdots, c_n \} \) choose a finite subset \( A = \{ b_1, \cdots, b_k \} \) to satisfy \( c(c_i A \sim A) < \delta c(A) \) for \( 1 \leq i \leq n \) where \( c(B) \) stands for the cardinality of \( B \) and \( \delta = \varepsilon(\max_{1 \leq i \leq n} ||f_i||)^{-1} n^{-1} \). Such \( A \) can be found by Følner’s characterization of discrete amenable groups [2] (see Namioka [9, p. 22]). Then for each \( 1 \leq i \leq n \)

\[
- \frac{1}{k} \sum_{j=1}^{k} l_{b_i} (f_i - c_i A) \leq c(c_i A - A) \left\| f_i \right\| /c(A) < \delta \left\| f_i \right\| \leq \varepsilon/n.
\]

Therefore \( |k^{-1} \sum l_{b_i} h_0(x)| \leq \varepsilon/2 \) loc. a.e. which finishes this proof.

**Remarks.** 1. It seems that this proposition does not hold true if \( G \) is not amenable as a discrete group (even in the case that \( G \) is compact).

2. If \( m(f) = \inf \{ \phi(f); \ \phi \in \text{LIM} \} \) then

\[
m(f) = -M(-f) = \sup_{\mathcal{A}} \left[ \text{ess inf} \frac{1}{n} \sum_{i} f(a_i, x) \right].
\]
3. One can show in a similar way that the support functional of the set of two-sided invariant means is $M_0f = \inf_{\mathcal{A}} \text{ess sup}_x (1/nm) \sum_{i,j} f(a_i, b_j)$ where $\mathcal{A}$ is the set of all pairs of finite tuples $(a_1, \ldots, a_n)(b_1, \ldots, b_m)$ of elements of $G$.

**Theorem 1.** Let $G$ be a locally compact $\sigma$-compact group which is amenable as a discrete group. If $\text{LIM} = \text{TLIM}$ then $G$ is discrete.

**Proof.** Assume that $G$ is not discrete and let $O$ be an open dense set in $G$ with $\lambda(O) < \frac{1}{2}$. Thus, if $G$ is not compact then $\phi(O) = 0$ for all $\phi \in \text{TRIM}$ hence $\Psi(O^{-1}) = 0$ for all $\Psi \in \text{TLIM}$ (see [4, p. 50]). If $G$ is compact then $\lambda(O^{-1}) = \lambda(O) < \frac{1}{2}$. Let $B = G^{-O^{-1}}$. Then $B$ is closed nowhere dense, $\Psi(B) = 1$ if $\Psi \in \text{TLIM}$ and $G$ is not compact while $\lambda(B) > \frac{1}{2}$ if $G$ is compact. (In this last case $\{\lambda\} = \text{TLIM} = \text{TRIM}$.)

We claim that $\phi(B) = 0$ for some $\phi \in \text{LIM}$. If not, then

$$m(1_B) = \inf_{\phi} \left\{ \phi(1_B); \phi \in \text{LIM} \right\} = \sup \text{ess inf}_{x} \frac{1}{n} \sum_{i=1}^{n} 1_B(a_i) = d > 0.$$ 

But then, there are $b_1, \ldots, b_k$ in $G$ such that $\text{ess inf}_{x} k^{-1} \sum_{i=1}^{k} 1_B(b_i, x) \geq d/2$ i.e. locally a.e. in $x$ one has $k^{-1} \sum_{i=1}^{k} 1_B(x) \geq d/2 > 0$. But this contradicts the fact that $A = G^{-\bigcup_{j=1}^{k} b_j^{-1}B}$ is open dense, hence of nonzero Haar measure and for $x \in A$, $k^{-1} \sum_{i=1}^{k} 1_B(b_i, x) = 0$. Using Remark 3 above one could easily show that in fact $\phi(B) = 0$ for some two sided invariant mean $\phi$ on $L^\infty(G)$.

**Remarks.** Let $G$ be a locally compact amenable group with $G_0 \subseteq G$ an open subgroup. Let $\lambda_0$ be the Haar measures on $G$ ($G_0$). As known and easily shown the $\lambda_0$ measurable sets comprise exactly the $\lambda$ measurable sets of $G$ which are included in $G_0$. We can and shall choose $\lambda_0$ to be the restriction of $\lambda$ to $G_0$. (We use the terminology of [7].)

For $f \in L^\infty(G)$ define $(\pi f)(x) = f(x)$ for $x \in G_0$. Then $\pi$ can be considered as a map onto $L^\infty(G_0)$. If $v \in L^\infty(G_0)^*$ is left invariant and $f \in L^\infty(G)$, let $(S_\alpha f)(z) = v(l_\alpha f)$ for all $z \in G$. Let $\{z_\beta G_0\}_{\beta \in I}$ be a fixed decomposition of $G$ into left cosets w.r.t. $G_0$. Then, the bounded function, $S_\alpha f$ is constant on each $z_\beta G_0$ (as known) since if $z = z_\alpha a$, $a \in G_0$ then $S_\alpha f(z_\alpha a) = v(l_{z_\alpha \alpha} f) = v(l_\alpha (l_{z_\alpha} f)) = S_\alpha f(z_\alpha)$, since $a \in G_0$. This implies that $S_\alpha f \in \text{UCB}_k(G)$ (i.e. is left uniformly continuous as in [7]) for all $f \in L^\infty(G)$ and left invariant $v \in L^\infty(G_0)^*$. This is the case since for all $z \in G$, $x \in G_0$, $S_\alpha f(zx) - S_\alpha f(z) = 0$ and $G_0$ is open.

Choose and fix now some $\text{LIM}$, $\mu_0$ on $C(G)$ and define for any left invariant $v \in L^\infty(G_0)^*$, $Tv v \in L^\infty(G)^*$, by $Tv(f) = \mu_0(S_\alpha f)$. 


As known and readily checked, $T$ maps the set of left invariant elements $\text{LIM}^\circ(G_0)$ into the set of left invariant elements $\text{LIM}^\circ(G)^\ast$. The above is a refinement of a construction due to M. M. Day [1, p. 533]. In the above context we have the

**Proposition 4.** Let $G$ be a locally compact amenable group and $G_0 \subseteq G$ an open normal subgroup.

If $Tv \in \text{TLIM}(G)$ for some $v \in \text{LIM}(G_0)$ then $v \in \text{TLIM}(G_0)$.

**Proof.** If $f \in L^\infty(G_0)$ denote by $f_1$ its $\{z_a\}$ periodic extension i.e. $f_1(z_axy)=f(x)$ for all $x \in G_0$ and all $a$. (Note that $\{z_a\}$ are fixed.) It is clear that $f_1$ is measurable (since it needs to be so only on compacta [7, p. 131], and $G_0$ is open).

If $z \in z_aG_0$ then

\[(*) \quad S_v(f_1)(z) = S_v(f_1)(z_a) = \nu(\pi l_{z_a}f_1) = \nu(f)\]

since if $x \in G_0$ then $(\pi l_{z_a}f_1)(x)=f_1(z ax)=f(x)$. Thus $(Tv)f_1 = \mu_0(S_v f_1) = \mu_0(\nu(f) \cdot 1_{G_0}) = \nu f$.

Fix now $\phi_0 \in \mathcal{P}(G)$ with support included in $G_0$. Then for $f \in L^\infty(G_0)$ and $x \in G_0$ one has:

\[
l_{z_a}(\phi_0 * f_1)(x) = \int f_1(y^{-1}z_ax)\phi_0(y) \, dy
\]

\[
= \int f_1((z_axyz_a^{-1})^{-1}z_ax)\phi_0(z_axyz_a^{-1}) \Delta(z_a^{-1}) \, dy
\]

\[
= \int_{G_0} f_1(z_axy^{-1}x)\phi_0(z_axyz_a^{-1}) \Delta(z_a^{-1}) \, dy
\]

\[
= \int_{G_0} f(y^{-1}x)\phi_0(z_axyz_a^{-1}) \Delta(z_a^{-1}) \, dy = (\Psi_a \otimes f)(x)
\]

where $\Psi_a(y) = \phi_0(z_axyz_a^{-1}) \Delta(z_a^{-1})$ for $y \in G_0$, thus $\Psi_a \in \mathcal{P}(G_0)$ and where $\otimes$ stands for convolution in $L_1(G_0)$. Note, that since $G_0$ is normal $\phi_0(z_axyz_a^{-1})$ has support included in $G_0$.

It follows that if $z \in z_aG_0$ then

\[
S_v(\phi_0 * f_1)(z) = S_v(\phi_0 * f_1)(z_a) = \nu(\pi l_{z_a}f_1) = \nu(\phi_0 \otimes f).
\]

Note that we have used in the last equality only the fact that $\nu \in \text{LIM}(G_0)$. From it alone, it follows (see Greenleaf [6, proof of Lemma 222, p. 27]) that $\nu(\phi \otimes f) = \nu(\Psi \otimes f)$ for all $\phi, \Psi \in \mathcal{P}(G_0)$.

Hence $Tv(\phi_0 * f_1) = \mu_0(S_v(\phi_0 * f_1)) = \nu(\phi_0 \otimes f)$. 
But by assumption $Tv \in TLIM$. Thus $Tv(\phi_0 \ast f_1) = (Tv)f_1 = vf$ and hence, for all $f \in L^\infty(G_0)$, $v(\phi_0 \otimes f) = v(f)$. The above remark implies that $v \in TLIM(G_0)$ and finishes this proof.

**Theorem 2.** Let $G$ be a locally compact group which is amenable as a discrete group. Assume that $G$ contains a $\sigma$-compact open normal subgroup. If $LIM(G) = TLIM(G)$ then $G$ is discrete.

**Remark.** 1. If $G$ has equivalent left and right uniform structures then $G$ contains a neighborhood $U$ of the identity with compact closure such that $xUx^{-1} = U$ for all $x \in G$. Thus $G_0 = \bigcup_{n=1}^{\infty} U^n$ is normal $\sigma$-compact and open. In particular the theorem certainly holds true for all locally compact abelian groups $G$. It also holds true for all $\sigma$-compact $G$ which are amenable as discrete groups (take $G = G_0$).

2. We could have assumed in this theorem that $G$ is a locally compact amenable group and the open normal $\sigma$-compact $G_0$ is amenable as discrete. This however readily implies that $G$ is amenable as discrete and we would not gain anything. (The discrete $G/G_0$ and $G_0$ with discrete topology are amenable hence so is $G$ with discrete topology.)

**Proof.** If $TLIM(G) = LIM(G)$ then $TLIM(G_0) = LIM(G_0)$ since $Tv \in TLIM(G) = LIM(G)$ for all $v \in LIM(G_0)$. Thus $v \in TLIM(G_0)$ by the previous proposition. We use now Theorem 1 and get that $G_0$ is discrete. Thus if $x \in G_0$, $\{x\}$ is open in $G_0$ hence in $G$. Hence $G$ is discrete.

The following is an interpretation of our and some known related results from the point of view of harmonic analysis on locally compact groups.

Let $H = \{H_x \}$ denote the linear span of $\{f - l_x f; f \in L^\infty(G), x \in G\}$. If $A \subseteq L^\infty(G)$ denote by $\tilde{A}$ its norm $[w^*]$ closure.

We need the following known remark whose proof uses a trick due to I. Namioka [9].

**Remark.** Let $\Psi$, $\Psi_1$, $\Psi_2 \in L^\infty(G)^*$, $\phi \in P(G)$ and define $(L_\phi \Psi f) = \Psi(\phi \ast f)$ for $f \in L^\infty(G)$. Let $\Psi_1 \vee \Psi_2 = \max(\Psi_1, \Psi_2)$ in the lattice $L^\infty(G)^*$ and $\Psi^+ = \Psi \vee O$, $\Psi^- = (-\Psi) \vee O$. If $\Psi \in L^\infty(G)^*$ satisfies $L_\phi \Psi = \Psi$ for all $\phi \in P(G)$, then so do $\Psi^+$ and $\Psi^-$. If $\phi \in P(G)$, $L_\phi (\Psi \vee O) \subseteq (L_\phi \Psi \vee L_\phi O) = \Psi \vee O = \Psi^+$. So $L_\phi \Psi^+ = \Psi^+$ and $\Psi^- \geq 0$. But $(L_\phi \Psi^+ - \Psi^+)(1) = 0$. Thus $L_\phi \Psi^+ = \Psi^+$. (Same true, if $L_\phi$ is replaced by $l_a^*$ for all $a \in G$.)

**Proposition 5.** (a) Let $G$ be compact and infinite. Then

$$\hat{A} \subset \hat{A}_c = \hat{A}_c^* = \hat{A}^* = \{f \in L^\infty(G); \lambda f = 0\}.$$

If $G$ is abelian (or even amenable as a discrete group) then $\hat{A} \neq \hat{A}_c$. 

Let \( G \) be a noncompact locally compact group. Then \( \hat{H} \subset \hat{H} \subset \hat{H} = L^\infty(G) \). Furthermore

(i) \( \hat{H} = L^\infty(G) \) iff \( \hat{H} = L^\infty(G) \) iff \( G \) is not amenable (i.e. \( \text{LIM} = \emptyset \)).

(ii) If \( G \) is \( \sigma \)-compact amenable then \( L^\infty(G)/\hat{H} \) is a nonseparable Banach space.

(iii) If \( G \) is \( \sigma \)-compact and amenable as discrete or amenable and containing such an open normal subgroup (in particular if \( G \) is locally compact abelian), then \( \hat{H} = \hat{H} \) iff \( G \) is discrete.

**Proof.** (a) \( \hat{H} \subset \hat{H} \) is due to the fact that \( \text{TLIM} \subset \text{LIM} \) [6, p. 25], the remark above and the Hahn-Banach theorem (this part with \( G \) not necessarily compact). Thus \( \hat{H} \subset \hat{H} \). If the inclusion were proper then there would exist some \( \phi \in L_1(G) \) such that \( \phi(H) = 0 \) but \( \phi(g) = 0 \) for some \( g \in H \). But then \( \phi \) is left invariant and in \( L_1(G) \) hence \( \phi = c\lambda \) for some scalar \( c \neq 0 \). Hence \( \phi(H) = \lambda(H) = 0 \) which cannot be. So \( \hat{H} \subset \hat{H} \subset \hat{H} = L^\infty(G); \lambda f = 0 \).

That \( \hat{H} = \{ f \in L^\infty(G); \lambda f = 0 \} \) is a consequence of Theorem 7.3, p. 360 of J. C. S. Wong [10] or can directly be proven. The rest of (a) is implied by the main theorem of this paper.

(b) If \( \hat{H} \neq L^\infty(G) \) there would exist \( 0 \neq \phi \in L_1(G) \) such that \( \phi(H) = 0 \). But then \( \phi \) is left invariant hence so are \( \phi^+ \), \( \phi^- \) and \( \phi^+ \neq 0 \) or \( \phi^- \neq 0 \). Assuming that \( \phi^+ \neq 0 \), \( \mu(A) = \int_A \phi^+ d\lambda \) is a measure on the Borel sets of \( G \) satisfying all the conditions in Hewitt-Ross [7, p. 194]. Hence \( \mu = c\lambda \) for some \( c > 0 \) (since \( \mu 
eq 0 \)).

Since \( \mu(G) < \infty \), \( \lambda(G) < \infty \) so \( G \) is compact. That (b)(i) holds is known and readily shown. (b)(ii) is shown as follows: If \( L^\infty(G)/\hat{H} \) would be separable there would exist a sequence \( \{ f_n \} \subset L^\infty(G) \) such that \( (f_n) + B \) is norm dense in \( L^\infty(G) \) (see [4, p. 63]). But \( \hat{H} = \{ f \in L^\infty(G); \Psi(f) = 0 \} \) for all \( \Psi \in \text{TLIM} \) Wong [10, p. 360]. Fix now some \( \Psi_0 \in \text{TLIM} \) and let \( \Psi_0 f_n = \alpha_n \). Then \( \{ \Psi_0 \} = \{ \Psi \in \text{TLIM}; \Psi f_n = \alpha_n, n \geq 1 \} \) since any \( \Psi \) which belongs to the right side will coincide with \( \Psi_0 \) on \( \hat{H} + B \) hence on \( L^\infty(G) \). We apply now [4, Theorem 5, p. 53] with \( K = \text{P}(G) \) hence \( A = \{ \Psi \in \text{TLIM}; \Psi(f_n - \alpha_n) = 0 \} = \{ \Psi_0 \} \) is norm separable. Thus \( G \) is compact. (b) (iii) is just our main theorem and the fact that \( \hat{H} = \hat{H} \) iff \( \text{LIM} = \text{TLIM} \) (by our remark above and the Hahn-Banach theorem).

**Main conjecture.** Let \( G \) be any amenable locally compact group. If \( G \) is noncompact then \( L^\infty(G)/\hat{H} \) is nonseparable. If \( G \) is nondiscrete then \( \hat{H} \) is nonseparable.

**Addition.** In the meantime W. Rudin sent us a preprint of a paper of his, in which he proves Theorem 2 without the assumption that (**) \( "G \) contains an open \( \sigma \)-compact normal subgroup\), but with the assumption...
that $G$ is amenable as discrete. His proof is different from ours and uses harmonic analysis type arguments. After reading his manuscript we found the following easy argument which removes the restriction (*).

**Proposition.** Let $G_0$ be an open noncompact subgroup of $G$, and

$$G = \bigcup_{x \in I} x_2 G_0, \quad x_2 G_0 \cap x_\beta G_0 = \emptyset \quad \text{if } \alpha \neq \beta.$$ 

If $A_0 \subseteq G_0$ is such that $\lambda(A_0) < \infty$ ($\lambda$—the Haar measure on $G$) then for all $\phi \in \text{TRIM}, \phi(\bigcup_{x \in I} x_2 A_0) = 0$.

**Proof.** Let $B_n \subset G_0$ be compact with $\lambda(B_n) = a_n \uparrow \infty$ and let $f_n = a_n^{-1} \mathbf{1}_{B_n}$, $A = \bigcup_{x} x_2 A_0$. Then

$$1_A * f_n^*(x) = a_n^{-1} \int 1_A(y) 1_{B_n}(y^{-1} x) \, dy$$

$$= a_n^{-1} \lambda(x B_n \cap A) \leq a_n^{-1} \lambda(x G_0 \cap A)$$

$$= a_n^{-1} \lambda(x_2 G_0 \cap A) = a_n^{-1} \lambda(A_0),$$

for some (hence all) $\alpha \in I$.

If $\phi \in \text{TRIM}$ then $\phi(A) = \phi(1_A * f_n^*) \leq a_n^{-1} \lambda(A_0) \rightarrow 0$.

**Remark.** If $\Psi \in \text{TLIM}$, then $\psi(\bigcup_{x} x_2 A_0^{-1} x_2^{-1}) = 0$. (See [4, pp. 49–50].)

To remove restriction (*) on $G$, let $G_0$ be any $\sigma$-compact, noncompact, open subgroup of $G$, if $G$ is noncompact, and $G = G_0$, if $G$ is compact. Let $A_0 \subseteq G_0$ be open dense with $\lambda(A_0) \leq \frac{1}{2}$ and $A = \bigcup_{x} A_0^{-1} x_2^{-1}$ ($x_2$ as above), $A = A_0$ if $G$ is compact. Let $B = G^{-1} A$. Then $\psi(B) = 1$ for all $\psi \in \text{TLIM}$, if $G$ is not compact, $\lambda(B) \geq \frac{1}{2}$ if $G$ is compact. $B$ is closed nowhere dense. Continue now as in the proof of Theorem 1.

**References**


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