ON AKCOGLU AND SUCHESTON'S OPERATOR CONVERGENCE THEOREM IN LEBESGUE SPACE

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Abstract. Let $T$ be a bounded linear operator on an $L_1$-space and $\tau$ its linear modulus. It is proved that if the adjoint of $\tau$ has a strictly positive subinvariant function then the following two conditions are equivalent: (i) $T^n$ converges weakly; (ii) $(1/n) \sum_{k=1}^{n} T^k$ converges strongly for any strictly increasing sequence $k_1, k_2, \cdots$ of nonnegative integers.

1. Introduction. Let $(X, \mathcal{M}, m)$ be a $\sigma$-finite measure space and $L_p(X) = L_p(X, \mathcal{M}, \mu)$, $1 \leq p \leq \infty$, the usual (complex) Banach spaces. If $A \in \mathcal{M}$ then $1_A$ is the indicator function of $A$ and $L_p(A)$ denotes the Banach space of all $L_p(A)$-functions that vanish a.e. on $X - A$. Let $T$ be a bounded linear operator on $L_1(X)$ and $\tau$ its linear modulus [2]. Thus $\tau$ is a positive linear operator on $L_1(X)$ such that

$$\|\tau\|_1 = \|T\|_1 \quad \text{and} \quad \tau g = \sup\{|Tf|; f \in L_1(X) \text{ and } |f| \leq g\}$$

for any $0 \leq g \in L_1(X)$. The adjoint of $T$ is denoted by $T^*$. Clearly $T$ is a contraction if and only if $\tau^* 1 \leq 1$. In [1] Akcoglu and Sucheston proved that if $T$ is a contraction then the following two conditions are equivalent: (i) $T^n$ converges weakly; (ii) $(1/n) \sum_{k=1}^{n} T^k$ converges strongly for any strictly increasing sequence $k_1, k_2, \cdots$ of nonnegative integers. In this note we shall prove that if $\tau^*$ has a strictly positive subinvariant function in $L_\infty(X)$ then the equivalence of (i) and (ii) still holds. Applying this result, we obtain that if $T$ is a positive linear operator on $L_1(X)$ such that $\sup_n \|(1/n) \sum_{k=0}^{n-1} T^k\|_1 < \infty$ and also such that $T^n f$ converges weakly for any $f \in L_1(X)$ with $\int f \, dm = 0$ and if $T^*$ has a strictly positive subinvariant function in $L_\infty(X)$, then for any $f \in L_1(X)$ with $\int f \, dm = 0$ and any strictly increasing sequence $k_1, k_2, \cdots$ of nonnegative integers, $(1/n) \sum_{k=1}^{n} T^k f$ converges strongly. This is a generalization of another result of Akcoglu and Sucheston [1].

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2. Results. Throughout this section we shall assume that there exists a strictly positive function \( s \in L_\infty(X) \) with \( \tau^*s \leq s \). In the proofs we shall also assume that \( (X, \mathcal{M}, m) \) is a finite measure space, since the \( L_1 \) of a \( \sigma \)-finite measure space is isometric to the \( L_1 \) of a finite measure space (cf. [1]).

**Theorem 1.** The following two conditions are equivalent:

(i) If \( f \in L_1(X) \) then \( T^nf \) converges weakly;

(ii) If \( f \in L_1(X) \) then \( \frac{1}{n} \sum_{k=1}^{n} T^k f \) converges strongly for any strictly increasing sequence \( k_1, k_2, \cdots \) of nonnegative integers.

**Proof.** We first prove that (i) implies (ii). For \( sf \in L_1(X) \), where \( f \in L_1(X) \), define \( V(sf) = sf \). Since \( \{ sf; f \in L_1(X) \} \) is a dense subspace of \( L_1(X) \) in the norm topology and \( ||V(sf)||_1 \leq ||sf||_1 \) (cf. [3]), \( V \) may be considered to be a linear contraction on \( L_1(X) \). Since \( V^n(sf) = T^n f \) for any \( n \geq 0 \) and \( T^nf \) converges weakly, it follows that \( V^n(sf) \) converges weakly. Thus, since \( V \) is a contraction, it is easily seen that for any \( A \in \mathcal{M} \) the limit \( \mu(A) = \lim_n \int_A V^n f \, dm \) exists. Since the measure \( m \) is finite, the Vitali-Hahn-Saks theorem implies that \( \mu \) is a countably additive measure on \( \mathcal{M} \) absolutely continuous with respect to \( m \). Therefore there exists a function \( g \in L_1(X) \) such that \( \mu(A) = \int_A g \, dm \) for any \( A \in \mathcal{M} \). It follows that \( V^n f \) converges weakly to \( g \). Thus, by Theorem 2.1 of [1], for any \( f \in L_1(X) \) and any strictly increasing sequence \( k_1, k_2, \cdots \) of nonnegative integers,

\[
\frac{1}{n} \sum_{i=1}^{n} V^{k_i}(sf) = \frac{1}{n} \left( \sum_{i=1}^{n} T^{k_i} f \right)
\]

converges strongly. Let \( \lim_n \frac{1}{n} \sum_{i=1}^{n} T^{k_i} f - f_0 = 0 \) for some \( f_0 \in L_1(X) \) and let \( \varepsilon > 0 \) be arbitrarily fixed. Since \( T^nf \) converges weakly, there exists a positive number \( \delta \) such that \( A \in \mathcal{M} \) and \( m(A) < \delta \) imply \( \int_A |T^n f| \, dm < \varepsilon \) for any \( n \geq 0 \). Choose \( \eta > 0 \) such that \( m(\{ x; s(x) < \eta \}) < \delta \) and \( \int_{\{x; s(x) < \eta\}} |f_0| \, dm < \varepsilon \), and put \( A = \{ x; s(x) < \eta \} \). Then

\[
\left\| \frac{1}{n} \sum_{i=1}^{n} T^{k_i} f - \frac{1}{m} \sum_{j=1}^{m} T^{k_j} f \right\|_1 \leq \left\| \frac{1}{n} \sum_{i=1}^{n} 1_A T^{k_i} f \right\|_1 + \left\| \frac{1}{m} \sum_{j=1}^{m} 1_A T^{k_j} f \right\|_1 + \left\| \frac{1}{n} \sum_{i=1}^{n} 1_{X-A} T^{k_i} f - \frac{1}{m} \sum_{j=1}^{m} 1_{X-A} T^{k_j} f \right\|_1 \leq 2 \varepsilon
\]

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and
\[ \left\| \frac{1}{n} \sum_{i=1}^{n} 1_{X-A} T^{k_i}f - 1_{X-A} \frac{1}{s} f_0 \right\|_1 \leq \frac{1}{\eta} \left\| \frac{1}{s} \left( \sum_{i=1}^{n} 1_{X-A} T^{k_i}f \right) - 1_{X-A} f_0 \right\|_1 \to 0 \]
as \( n \to \infty \), from which we observe that \( \frac{1}{n} \sum_{i=1}^{n} T^{k_i}f \) is a Cauchy sequence in \( L_1(X) \), and hence \( \frac{1}{n} \sum_{i=1}^{n} T^{k_i}f \) converges strongly.

Conversely if (ii) holds, then it follows easily that \( \sup_n \| T^n \|_1 < \infty \) and that for any \( f \in L_1(X) \) and any \( A \in \mathcal{A} \), \( \lim_n \int_A T^n f \, dm \) exists, and hence \( T^n f \) converges weakly. This completes the proof of Theorem 1.

**Theorem 2.** Let \( T \) be a positive linear operator on \( L_1(X) \) with
\[ \sup_n \left\| \frac{1}{n} \sum_{k=0}^{n-1} T^k \right\|_1 < \infty \]
and suppose \( T^* s \leq s \) for some \( 0 < s \in L_\infty(X) \). Then the following two conditions are equivalent:

(i) If \( f \in L_1(X) \) and \( \int f \, dm = 0 \), then \( T^n f \) converges weakly;

(ii) If \( f \in L_1(X) \) and \( \int f \, dm = 0 \), then for any strictly increasing sequence \( k_1, k_2, \ldots \) of nonnegative integers, \( \frac{1}{n} \sum_{i=1}^{n} T^{k_i}f \) converges strongly.

**Proof.** Suppose (i) holds. It is known [3] that if \( T \) has no nontrivial nonnegative invariant function in \( L_1(X) \), then the operator \( V \) introduced above also has no nontrivial nonnegative function in \( L_1(X) \). Thus it follows from [1] that, if \( T^n f \) converges weakly then
\[ \lim_n \| V^n(sf) \|_1 = \lim_n \| s T^n f \|_1 = 0. \]

Let \( \epsilon > 0 \) be arbitrarily fixed, and let \( \delta \) be a positive number such that \( A \in \mathcal{A} \) and \( m(A) < \delta \) imply \( \int_A |T^n f| \, dm < \epsilon \) for any \( n \geq 0 \). Choose \( \eta > 0 \) such that \( m(\{ x ; s(x) < \eta \}) < \delta \), and put \( A = \{ x ; s(x) < \eta \} \). Then
\[ \| T^n f \|_1 \leq \| 1_A T^n f \|_1 + \eta^{-1} \| 1_{X-A} s T^n f \|_1 \]
\[ < \epsilon + \eta^{-1} \| s T^n f \|_1 \]
and \( \| s T^n f \| \to 0 \) as \( n \to \infty \), thus \( \lim_n \| T^n f \|_1 = 0. \)

If there exists \( 0 \leq h \in L_1(X) \) with \( \| h \|_1 > 0 \) and \( Th = h \), then it follows from [1] that for any \( f \in L_1 \), \( T^n f \) converges weakly. Thus the strong convergence of \( \frac{1}{n} \sum_{i=1}^{n} T^{k_i}f \) for any strictly increasing sequence \( k_1, k_2, \ldots \) of nonnegative integers follows from Theorem 1.

Clearly (ii) implies (i), and the proof is complete.
BIBLIOGRAPHY


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