PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 40, Number 2, October 1973

PROJECTIVITY OF THE WHITEHEAD SQUARE

DUANE RANDALL

ABSTRACT. We show that the Whitehead square of the generator of $\pi_n(S^n)$ is not a projective homotopy class for any integer n with neither n or $n+1$ being a power of 2.

1. **Introduction.** Let $\omega_n \in \pi_{2n-1}(S^n)$ denote the Whitehead square of the generator of $\pi_n(S^n)$. To determine the integers n for which ω_n can be halved is an important problem in homotopy theory. A homotopy class $[f] \in \pi_m(X)$ is called projective in [7] iff f is homotopic to $g \circ \pi$ for some map $g: P^m \to X$ where $\pi: S^m \to P^m$ denotes the double covering. We remark that ω_n can be halved iff there exists $g: P^{2n-1} \to S^n$ with $[g \circ \pi] = \omega_n$ and $[g \circ i] = 0$ where $i: P^{2n-2} \subset P^{2n-1}$. In this note we shall prove the following.

THEOREM. *ω_n is not projective for any positive integer n with neither n nor $n+1$ a power of* 2.

The proof consists of constructing a complex with a nontrivial cup product mod 2 if ω_n is projective and then applying decomposability of Steenrod squares to obtain a contradiction.

Note that $\omega_n = 0$ for $n = 1, 3, 7$ and so is trivially projective. By [5] ω_4 is projective, but ω_4 can not be halved. Since $2\pi_m(X)$ consists of projective classes for odd m and any space X by [7], ω_{15} and ω_{31} can be halved (see [3]) and so are projective.

2. **Notation.** The coefficient group for cohomology is Z_2 whenever omitted. We write $H^*(P^\infty) = Z_2[\alpha]$ and recall $\mathrm{Sq}^i \alpha^j = \binom{j}{i} \alpha^{i+j}$. The following result on binomial coefficients is needed.

LEMMA. *Let $n = 2s + 2^t$ with $0 < s < 2^{t-1}$. If $\binom{2^t-1-i}{2s-2i}$ is odd with $0 < i \leq s$, then $\binom{n+i-1}{n-i}$ is even.*

PROOF. Let $\alpha(m)$ denote the number of 1's in the dyadic expansion of m. Now $\alpha(2s-i) = \alpha(2s-2i) + \alpha(i)$ since the dyadic expansions of i and $2s-2i$ are disjoint by hypothesis. If $\binom{n+i-1}{n-i}$ is odd, $\alpha(2s+i-1) = \alpha(2s-i) + \alpha(2i-1)$. Thus $\alpha(2s+i-1) = \alpha(2s-2i) + \alpha(i) + \alpha(2i-1)$, a contradiction.

Received by the editors January 17, 1973 and, in revised form, February 12, 1973.
AMS (MOS) subject classifications (1970). Primary 55E15, 55E40; Secondary 55G10.
Key words and phrases. Whitehead product, projective homotopy class, Steenrod square, Hopf construction.

3. Proof. Assume ω_n is projective. We obtain the following diagram of horizontal and vertical Puppe sequences with homotopy commutative squares.

$$\begin{array}{ccccccc} S^{2n-1} & \xrightarrow{f} & S^n & \longrightarrow & S^n \cup_f e^{2n} & \longrightarrow & S^{2n} \\ \downarrow{\scriptstyle \pi} & & \| & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \Sigma\pi} \\ P^{2n-1} & \xrightarrow{g} & S^n & \xrightarrow{j} & C_g & \xrightarrow{l} & \Sigma P^{2n-1} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ P^{2n} & \longrightarrow & * & \longrightarrow & \Sigma P^{2n} & \longrightarrow & \Sigma P^{2n} \end{array} \tag{3.1}$$

Here $[f]=\omega_n$. C_g denotes the mapping cone of g which exists by hypothesis.

Theorem A. *ω_n is not projective for $n=2s+2^t$ with $0<s<2^{t-1}$.*

Proof. Assume ω_n projective and let u generate $H^n(C_g; Z)=Z$ in (3.1). Note j^*u generates $H^n(S^n; Z)$ and h^*u generates $H^n(S^n \cup_f e^{2n}; Z)$. Now ω_n has Hopf invariant ± 2 and $\Sigma\pi$ induces multiplication by 2 on integral cohomology in dimension $2n$. Thus $u \cup u$ generates $H^{2n}(C_g; Z)$ so $\mathrm{Sq}^n u \neq 0$. By [1]

$$\mathrm{Sq}^n = \mathrm{Sq}^{2s}\mathrm{Sq}^{2^t} + \sum_{i=1}^{s} \binom{2^t-1-i}{2s-2i} \mathrm{Sq}^{n-i}\mathrm{Sq}^i. \tag{3.2}$$

If $\binom{2^t-1-i}{2s-2i}$ is odd, $\mathrm{Sq}^{n-i}(\Sigma\alpha^{n+i-1})=\binom{n+i-1}{n-i}\Sigma\alpha^{2n-1}=0$ by the Lemma so $\mathrm{Sq}^{n-i}\mathrm{Sq}^i u=0$. Also $\mathrm{Sq}^{2s}(\Sigma\alpha^{n+2^t-1})=0$ so $\mathrm{Sq}^n u=0$, contradiction.

Theorem B. *ω_n is not projective for any odd integer n with $n+1$ not a power of 2.*

Proof. Assume ω_n projective so (3.1) exists and write $m=n+1=2s+2^t$ with $0<s<2^{t-1}$. Note that $H^n(C_g; Z)=Z\oplus Z_2$. Let ψ: $S^n \times S^n \to S^n \cup_f e^{2n}$ be the natural map identifying axes. Set $F=h\circ\psi$. The morphism F^*: $H^{2n}(C_g; Z)\to H^{2n}(S^n\times S^n; Z)$ is multiplication by 2 and so F^* is trivial with Z_2 coefficients in dimension $2n$. The Hopf construction applied to F gives a map $\rho: S^{2n+1}\to \Sigma C_g$ with mapping cone X. Sq^m is nontrivial on $H^m(X)$ by [4]. (See [6] for a complete proof.) Consider the following diagram of cofibrations and homotopy commutative squares.

$$\begin{array}{ccccc} S^{2m-1} & \xrightarrow{\rho} & \Sigma C_g & \longrightarrow & X \\ \| & & \downarrow{\scriptstyle \Sigma l} & & \downarrow{\scriptstyle r} \\ S^{2m-1} & \xrightarrow{\Sigma^2\pi} & \Sigma^2 P^{2n-1} & \longrightarrow & \Sigma^2 P^{2n} \end{array} \tag{3.3}$$

We now apply (3.2). Note that Sq^i is trivial on $H^m(X)$ for odd i since $\mathrm{Sq}^i=\mathrm{Sq}^1\mathrm{Sq}^{i-1}$ and $\mathrm{Sq}^1(\Sigma^2\alpha^{m+i-3})=0$. Suppose $\binom{2^t-1-i}{2s-2i}$ is odd with i even. We must show $\mathrm{Sq}^{m-i}(\Sigma^2\alpha^{m+i-2})=0$ so that $\mathrm{Sq}^{m-i}\mathrm{Sq}^i$ is trivial on $H^m(X)$ via r^* in (3.3). In $H^*(P^\infty)$ $\mathrm{Sq}^{m-i}(\alpha^{m+i-2})\cup\alpha=\mathrm{Sq}^{m-i}(\alpha^{m+i-1})=0$ by the Lemma so $\mathrm{Sq}^{m-i}(\alpha^{m+i-2})=0$. Also $\mathrm{Sq}^{2s}(\alpha^{m+2^t-2})=0$ so Sq^m is trivial on $H^m(X)$, contradiction.

References

1. J. Adem, *The iteration of the Steenrod squares in algebraic topology*, Proc. Nat. Acad. Sci. U.S.A. **38** (1952), 720–726. MR **14**, 306.

2. I. M. James, *On the homotopy type of Stiefel manifolds*, Proc. Amer. Math. Soc. **29** (1971), 151–158. MR **43** #1184.

3. ———, *Note on Stiefel manifolds*. I, Bull. London Math. Soc. **2** (1970), 199–203. MR **41** #9286.

4. ———, *On the decomposability of fibre spaces*, The Steenrod Algebra and its Applications (Proc. Conf. to Celebrate N. E. Steenrod's 60th Birthday, Battelle Memorial Inst., Columbus, Ohio, 1970), Lecture Notes in Math., vol. 168, Springer, Berlin, 1970. MR **43** #4038.

5. E. Rees, *Symmetric maps*, J. London Math. Soc. (2) **3** (1971), 267–272. MR **43** #6923.

6. E. Thomas, *On functional cup-products and the transgression operator*, Arch. Math. **12** (1961), 435–444. MR **26** #6972.

7. P. Zvengrowski, *Skew linear vector fields on spheres*, J. London Math. Soc. **3** (1971), 625–632.

Department of Mathematics, University of Notre Dame, Notre Dame, Indiana 46556

Current address: Pontificia Universidade Católica, Rio de Janeiro, Brazil