METRIC INEQUALITIES AND THE ZONOID PROBLEM

H. S. WITSENHAUSEN

Abstract. For normed spaces the hypermetric and quasihypermetric properties are equivalent and imply the quadrilateral property. The unit ball of a Minkowski space is a zonoid if and only if the dual space is hypermetric. The unit ball of \( L_p^n \) is not a zonoid for \( n=3, \, p<\log 3/\log 2 \), and for \( p\leq 2-(2n \log 2)^{-1}+o(n^{-1}) \). The elliptic spaces \( \delta^n, d>1 \), are not quasihypermetric.

A metric space \((S, d)\) is said to be hypermetric (Kelly [3]) when

\[
\sum_{i,j=1}^{n} w_i w_j d(x_i, x_j) \leq 0
\]

for all \( n>0, \, x_1, \cdots, x_n \) in \( S \), and \( w_1, \cdots, w_n \) integers with sum 1. This implies [5] that (1) also holds for real \( w_i \) of sum 0, which is called the quasihypermetric property.

A piecewise linear inequality (PLI) is a relation of the form

\[
\sum_{i=1}^{k} c_i \left| \sum_{j=1}^{n} a_{ij} x_j \right| \geq 0
\]

which holds for all \( n \)-tuples \( x_1, \cdots, x_n \) of real numbers, with fixed real \( c_i \) and \( a_{ij} \). An example is the quadrilateral inequality [8]

\[
|x| + |y| + |z| - |x + y| - |y + z| - |z + x| + |x + y + z| \geq 0.
\]

Since the real line is hypermetric [4], (1) generates an infinite family of PLI’s of the form

\[
\sum_{i,j=1}^{n} (-w_i w_j) |x_i - x_j| \geq 0
\]

for \( w_i \) integers of sum 1 and for \( w_i \) reals of sum 0.

The PLI (2) is said to extend to the normed space \( N \) if it holds with the absolute value function replaced by the norm and \( x_1, \cdots, x_n \) arbitrary elements of \( N \).

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A zonoid [1], [2] is a convex body belonging to the closure (in the Hausdorff set metric) of the class of zonotopes (polytopes which are Minkowski sums of segments).

The theorems of I. J. Schoenberg [7] and P. Lévy [6] imply that the above concepts are related.

**Proposition 1.** For a real normed space $N$ the following 3 properties are equivalent.

(i) every PLI extends to $N$,
(ii) $N$ is quasihypermetric,
(iii) $e^{-\|x\|}$ is positive definite on $N$.

**Proof.** If every PLI extends to $N$ then in particular $N$ is quasihypermetric. Following an argument of Schoenberg, consider $n+1$ points $x_0, \ldots, x_n$ in $N$ with weights $-\sum_{i=1}^{n} w_i, w_1, \ldots, w_n$ where the $w_i$ are arbitrary reals. This yields

$$\sum_{i,j=1}^{n} w_i w_j (\|x_i - x_0\| + \|x_j - x_0\| - \|x_i - x_j\|) \geq 0,$$

that is, the parenthesis is positive definite. Then its exponential is positive definite and, absorbing $e^{\|x_i - x_0\|}$ into $w_i$, $e^{-\|x\|}$ is shown to be positive definite. Conversely, if $e^{-\|x\|}$ is positive definite on $N$ it is positive definite on every finite dimensional subspace of $N$. By Lévy's theorem [6], [1] these subspaces are isometrically isomorphic to subspaces of $L_1(0, 1)$ to which any PLI extends by integration. Since each PLI involves only finite systems of vectors, it extends to all of $N$.

In [4] Kelly raised the question of the possible relations between the hypermetric and quadrilateral properties in normed spaces. Applying Proposition 1 one has

**Corollary 1.1.** For real normed spaces, the hypermetric and quasihypermetric properties are equivalent and they imply the quadrilateral property.

For $1 \leq p \leq 2$, $e^{-\|x\|}$ is known [7] to be positive definite on $L_p(0, 1)$, hence

**Corollary 1.2.** $L_p(0, 1)$ (and a fortiori $l_p$) is hypermetric and quadrilateral for $1 \leq p \leq 2$.

This had been conjectured by Kelly [4] and the Smileys [8]. For finite dimensional real normed spaces (Minkowski spaces) the positive definiteness of $e^{-\|x\|}$ is equivalent [1] to the property that the unit ball of the dual space is a zonoid. Thus one has

**Corollary 1.3.** The unit ball of a Minkowski space is a zonoid if and only if the dual space is hypermetric.
Thus the known fact that all Minkowski planes are hypermetric \([4]\) follows from the elementary fact that all centrally symmetric convex polygons are sums of segments.

For \(n \geq 3\), let \(p_n\) be the smallest \(p\) such that the unit ball of \(l_p^n\) is a zonoid. One has \(p_3 \leq p_n \leq p_{n+1} \leq 2\), Bolker \([1]\), \([2]\) has conjectured that \(p_3 = 2\). He reports the following bounds of Rosenthal: \(p_3 > \log 9 / \log 7\) and \(p_n > 2 \log n / \log 3n\), hence \(p_n \geq 2 - \log 9 / \log n + o((\log n)^{-1})\).\(^1\) These bounds can be substantially improved.

**PROPOSITION 2.** One has \(p_3 \geq \log 3 / \log 2\) and \(p_n \geq 2 - 1/2n \log 2 + o(n^{-1})\).

**Proof.** For \(n = 3\), \(p < \log 3 / \log 2\) the quadrilateral inequality in the dual space is violated for \(x = (1, 1, -1), y = (1, -1, 1), z = (-1, 1, 1)\), as observed by the Smileyes \([8]\).\(^2\) For large even \(n = 2m\), consider the quasi-hypermetric inequality in the dual \(l_q^{2m}\), with \(w_i = 1\) at the \(2^m\) points with the first \(m\) coordinates equal to \(\pm 1\) and the last \(m\) coordinates 0, \(w_i = -1\) at the \(2^m\) points with first \(m\) coordinates 0 and the last \(m\) equal to \(\pm 1\). All distances between the two sets are \((2m)^{1/q}\) while distances within each set are of the form \(2k^{1/q}\) with \(0 \leq k \leq m\). Counting the number of occurrences of each distance, a violation of the inequality is seen to require

\[
2 \left(2^{m-1} \sum_{k=0}^{m} \binom{m}{k} (2k^{1/q}) \right) > 2^{2m}(2m)^{1/q}
\]

or \(2E\{k^{1/q}\} > (2m)^{1/q}\) with \(k\) binomially distributed. For large \(m\), expand \(k^{1/q}\) about the mean \(k = m/2\) and let \(1/q = \frac{1}{2} + \epsilon\). Then the violation occurs for \(\epsilon < -(16m \log 2)^{-1} + o(m^{-1})\), so that \(p_n \geq 2 - (2n \log 2)^{-1} + o(n^{-1})\) as claimed.

Kelly \([3]\) has shown that spherical spaces are hypermetric. This no longer holds when antipodes are identified.

**PROPOSITION 3.** The elliptic plane \(S^2\) is not quasihypermetric.

**Proof.** Assume the opposite, and consider the function, defined for \(\mu \in C(S^2)^*\), by \(F(\mu) = \int \mu(dx) \int \mu(dy) xy\), where \(xy\) is the elliptic distance and the integrals range over the compact space \(S^2\). By (1) the function is nonpositive, hence concave on the subspace \(\{\mu | \int \mu(dx) = 0\}\). The concavity holds as well on the parallel subspace \(\{\mu | \int \mu(dx) = 1\}\) and in particular on the set \(P\) of probability measures on \(S^2\). For \(\mu\) in \(P\) and \(\tau\) in the compact group \(G\) of isometries of \(S^2\), let \(\mu^*\) be the mixture of the displaced measures \(\mu \circ \tau\) under normalized Haar measure on \(G\). Then \(\mu^*\) is the uniform

\(^1\) Thus \(L_p(0, 1)\) is not hypermetric for \(p > 2\).

\(^2\) Alternatively, the hypermetric inequality is violated for the choice of \(w_i = 1\) at \((\pm 1, \pm 1, 0)\) and \(w_i = -1\) at \((0, 0, 0), (0, 0, \pm 1)\).
distribution on $S^2$ and by concavity $F(\mu^*) \geq F(\mu)$. However, the distribution $\mu$ assigning equal probabilities to the vertices of an equilateral triangle of side length $D$, the diameter of $S^2$, yields $F(\mu) = 2D/3$ while $F(\mu^*) = 2D/\pi$, a contradiction.

That $S^2$ is not hypermetric already follows from the violation of the hypermetric inequality that occurs for the choice of $w_i = -1$ at 3 mutually orthogonal lines and $w_i = +1$ at their 4 trisectors.

Since $S^2 \subset S^d$ for $d > 2$ one has

**Corollary 3.1.** For $d > 1$ the elliptic space $S^d$ is not quasihypermetric.

**References**


Bell Telephone Laboratories, Inc., Murray Hill, New Jersey 07974