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HOCHSCHILD DIMENSION OF A SEPARABLY GENERATED FIELD

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ABSTRACT. Let K be an $\aleph_k$-generated field extension of the field F with transcendence degree n. Set bidim(K)=the projective dimension of K as a $K \otimes_F K$-module. Then K locally separably generated implies bidim$(K) \leq k+n+1$, and K separably generated implies bidim$(K)=k+n+1$.

Let F be a commutative ring, K an F-algebra. The Hochschild or bidimension of K over F, written bidim(K), is the projective dimension of K as a module over the ring $R=K \otimes_F K^{\mathrm{op}}$. In **[3]**, Hochschild showed that if K is finite over a field F, then bidim$(K)=0$ if and only if K is separable over F, i.e., for all field extensions L of F, $K \otimes_F L$ is semisimple. Noether **[8]** had done this for commutative K earlier. Rosenberg and Zelinsky **[12]** showed bidim$(K)=0$ implies $[K:F]<\infty$. They also obtained results in the case K is a field relating bidim(K) and the transcendence degree of K over F. MacRae **[6]** took their work and characterized all fields K of bidimension 1 over the field F by showing they are countably generated and so, by Rosenberg and Zelinsky, either separable algebraic extensions of dimension $\aleph_0$ or finite separable extensions of rational function fields in one variable. The purpose of this note is to compute the bidimension of any separably generated extension field K of the field F. If K is not locally separably generated, Rosenberg and Zelinsky showed bidim(K) is infinite, but only an upper bound will be obtained in this paper on the locally separably generated case.

Let $\aleph_{-1}$ denote any finite cardinal and $\aleph_\infty$ any cardinal $\geq \aleph_\omega$. $\#B$ will denote the cardinality of the set B, and $[K:F]$ the dimension of the vector space K over the field F. We show the

THEOREM. *Let K be a field extension of the field F with separating transcendence basis B. Let $[K:F(B)]=\aleph_m$. Then* bidim$(K)=\#B+m+1$.

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The method of proof is based on the computation in **[10]** or **[11]** of the global dimension of a direct product of fields, but with an appropriate substitute for the "nice sets of idempotents" used in that case.

1. **Elementary remarks.** Throughout the rest of this paper, F will be a fixed field, and K an extension field of F. All fields will be subfields of K containing F. $\otimes$ will mean $\otimes_F$, R will denote the ring $K\otimes K$, and I_K will denote the kernel of the multiplication map $R\to K$. Then, since $0\to I_K\to R\to K\to 0$ is exact, $\text{bidim}(K)\neq 0$ implies $\text{bidim}(K)=\text{p.d.}(I_K)+1$. The reader is referred to any standard reference, such as **[2]** or **[5]**, for the definition of projective dimension. Hence we need only calculate $\text{p.d.}(I_K)$ to get our result. We will do that in the algebraic case. Our initial reduction is to that case.

1. LEMMA. *Let S be any ring, x a central nonunit, nonzero divisor of S.* Let $\bar{S}=S/xS$. *If $M_{\bar{S}}$ is an $\bar{S}$-module of finite projective dimension, then* $\text{p.d.}_S(M)=\text{p.d.}_{\bar{S}}(M)+1$.

PROOF. See **[4]** or perhaps **[11]**.

2. LEMMA. *Let $x\in K$ be transcendental over F. Then*

$$K\otimes_F K/(x\otimes 1-1\otimes x)\simeq K\otimes_{F(x)} K$$

and $\bar{x}=x\otimes 1-1\otimes x$ is not a zero divisor in $K\otimes_F K=R$.

PROOF. The natural map $R\to K\otimes_{F(x)} K$ has kernel Z generated by

$$\{up\otimes v-u\otimes pv \mid u,v\in K, p\in F(x)\}.$$

But $up\otimes v-u\otimes pv=(p\otimes 1-1\otimes p)(u\otimes v)$, so $\{p\otimes 1-1\otimes p \mid p\in F(x)\}$ generates Z. In particular, $\bar{x}\in Z$. Now let $p/q\otimes 1-1\otimes p/q$ be a generator of Z, with $p, q\neq 0\in F[x]$. Let $\equiv$ denote congruence modulo $(\bar{x})$. Then $x\otimes 1\equiv 1\otimes x$, so $p\otimes 1\equiv 1\otimes p$ and $q\otimes 1\equiv 1\otimes q$. Then $(q\otimes 1)^{-1}=1/q\otimes 1\equiv(1\otimes q)^{-1}=1\otimes 1/q$, so $p/q\otimes 1\equiv 1\otimes p/q$, and $Z=(\bar{x})$. Let $0\neq\sum u_i\otimes v_i\in R$. If $\sum u_i\otimes xv_i=\sum xu_i\otimes v_i$, then for all i, $xv_i=\sum\alpha_{ij}v_j$ with $\alpha_{ij}\in F$, so $\det(xI-(\alpha_{ij}))=0$, contradicting the transcendental property of x.

3. LEMMA. *Let K be a field extension of F with separating transcendence basis B and $[K:F(B)]=\aleph_m$. If $\#B=0$ implies* $\text{bidim}(K)=m+1$, *then for any K,* $\text{bidim}(K)=\#B+m+1$.

PROOF. If both $\#B$ and m are finite, we use induction on $n=\#B$. The basis is assumed. If $\#B>0$, let $x\in B$. Then K is a $K\otimes K/(x\otimes 1-1\otimes x)\simeq K\otimes_{F(x)} K$-module of projective dimension $(n-1)+m+1$ by the induction hypothesis. Apply Lemma 1.

If $n=\#B$ or m is infinite, then, given any natural number k, K contains a purely transcendental subfield of degree $k+1$ and hence bidimension $k+1$ by repeated applications of Lemma 1, or K contains an $\aleph_k$-dimensional extension of $F(B)$ which has bidimension $n+k+1$ by the finite case. In either case, since Rosenberg and Zelinsky show in **[12]** that the bidimension of K is equal to or greater than the bidimension of any subfield, $\text{bidim}(K)=\infty=n+m+1$.

We may thus assume for the rest of this paper that K is a separable algebraic extension of F.

4. Remark. Let L be a finite dimensional separable extension of F. Then the number of subfields of L containing F is finite. For a proof, pick a book—any book—on classical Galois theory.

5. Corollary. *$[K:F]=\aleph_n$ implies K contains (at most) $\aleph_n$ finite dimensional extensions of F.*

Proof. K may be expressed as an ascending union of $\aleph_n$ smaller dimensional subfields, and every finite extension of F is in one of these. Hence K has at most $\aleph_n \cdot \aleph_n$ finite dimensional subfields by induction on n. It is clear this number is reached.

6. Remark. Let α be any element of K. Set $\bar{\alpha}=\alpha\otimes 1-1\otimes\alpha$. Let $0\neq x=\sum u_i\otimes v_i$. The set $\{v_i\}$ will be called the "second coordinates" of x. Without loss of generality, these "second coordinates" are linearly independent over F. Then $\bar{\alpha}x=0$ implies $\{v_i\}\cup\{\alpha v_i\}$ must be linearly dependent, so there exist $\{k_i, l_i\}\subseteq F$ not all zero with $\sum k_iv_i+\alpha\sum l_iv_i=0$. Since the v_i are linearly independent, $\sum l_iv_i\neq 0$, so $\alpha=\sum k_iv_i/\sum l_iv_i \in F[\{v_i\}]$.

Now let us look at I_K. For any finite extension L of F in K, $I_L=\text{kernel}(L\otimes L\to L)$ is generated by an idempotent e_L by the Noether-Hochschild result mentioned in the introduction ($L\otimes L\to L\to 0$ splits). Any element in $I_K\cap L\otimes L$ must lie in e_LR. This use of L for finite dimensional extensions of F in K and e_L for generating idempotents in $I_L=I_K\cap L\otimes L$ will be maintained throughout the paper, as will the notation $\bar{\alpha}=\alpha\otimes 1-1\otimes\alpha$ for all $\alpha\in K$.

7. Lemma. *Let I be any right ideal of a ring S such that I is generated by commuting idempotents $\{e_j \mid j\in J\}$. Then $\#J=\aleph_n\Rightarrow \text{p.d.}(I)\leq n$.*

Proof. If $n=0$, $I=\bigoplus_{i=0}^{\infty} e_i\prod_{j=0}^{i-1}(1-e_j)S$ is projective. Auslander's Proposition 1 of **[1]** and an easy induction complete the proof (see **[10]** or **[11]**).

8. Corollary. $[K:F]=\aleph_n\Rightarrow \text{p.d.}(I_K)\leq n$.

Proof. Any element $\sum u_i\otimes v_i$ of I_K lies in $L\otimes L$ for some finite dimensional L. Hence $\{e_L\}$ generate I_K. Apply 5 and 7.

2. **The lower bound.** To get a lower bound on the bidimension of K, we need a lower bound on the dimension of I_K, still assuming K is algebraic. The machinery of **[10]** can be used for getting a projective resolution and direct summand of a projective image in that resolution to apply induction. However, the idempotents generating I_K are not "nice" in the sense of **[10]**, so a different basis and induction step are needed. Trying to use induction on the dimension of K over F directly did not seem to work, but led to the following type of ideal which does.

Let Λ be a set of finite dimensional fields L directed under $\subseteq$. Let f be an idempotent of R such that, for all $L \in \Lambda$, $0 \neq f(1-e_L)$. Set $I_f(\Lambda) = \sum_{L\in\Lambda} fe_L R$. Note that if Λ is the set of all finite dimensional extensions of F in K and K is infinite dimensional, then $\#\Lambda = [K:L]$, $1 \cdot (1-e_L) \neq 0$ for all $L \in \Lambda$, and $I_1(\Lambda) = I_K$.

9. PROPOSITION. *For any idempotent f and any Λ such that for all $L \in \Lambda$, $0 \neq f(1-e_L)$, we have*

$$\text{p.d.}(I_f(\Lambda)) \leqq n \Rightarrow \#\Lambda \leqq \aleph_n.$$

PROOF. We use induction on n. The basis is a modification of an argument in Magid [7]. Let $n=0$, so $I_f(\Lambda)$ is projective. By the dual basis lemma (the map $\bigoplus_{L\in\Lambda} R \to \sum fe_L R \to 0$ splits) there exist maps

$$\{\theta_L \in \text{Hom}_R(I_f(\Lambda), R) \mid L \in \Lambda\}$$

such that for all $u \in I_f(\Lambda)$, $u = \sum_L \theta_L(u) fe_L$ and $\theta_L(u) = 0$ for almost all $L \in \Lambda$. Since θ_L is an R-map, $\theta_L(u)fe_L = \theta_L(ufe_L) = u\theta_L(fe_L)$. Set $g_L = \theta_L(fe_L)$. Then $ug_L = 0$ for almost all L, and $g_L = fg_L$ for all $L \in \Lambda$.

Case (i). $g_L = 0$ for almost all L. Then $I_f(\Lambda) = \sum_{i=1}^n fe_{L_i} R$, and if L is an element of Λ containing each L_i, $I_f(\Lambda) = e_L fR$. Now let $\alpha \in \bigcup_{L\in\Lambda} L$. Then $f\bar{\alpha} \in fe_L R$, so $f\bar{\alpha} = fe_L\bar{\alpha}$, i.e. $(1-e_L)f\bar{\alpha} = 0$. By Remark 6, α must belong to the finite dimensional subfield L' generated by "second coordinates" of the nonzero element $(1-e_L)f$. Hence $\bigcup_{L\in\Lambda} L \subseteq L'$ which has only finitely many subfields, and so $\#\Lambda < \aleph_0$.

Case (ii). There exists a countable set $\{L_i \mid i \in \omega\} \subseteq \Lambda$ such that for all $i \in \omega$, $g_{L_i} \neq 0$. Let L' be the subfield of K generated by all "second coordinates" of $\{g_{L_i} \mid i \in \omega\}$. Then $[L':F] = \aleph_0$ so L' has only $\aleph_0$ finite dimensional subfields by 5. For all $\alpha \in \bigcup_{L\in\Lambda} L$, $\bar{\alpha} fg_{L_i} = 0$ for almost all i, so by Remark 6, $\alpha \in L'$ and $\bigcup_{L\in\Lambda} L \subseteq L'$. Hence $\#\Lambda = \aleph_0$.

For $n > 0$, the machinery of **[10]** is available to give our induction step. Since it is written out in reasonable detail in **[10]** and **[11]**, we just sketch this machinery.

Assume $\#\Lambda > \aleph_n$, and linearly order Λ in such a way that no ordinal

$\leqq\aleph_n$ is cofinal in Λ. One gets a projective resolution of $I_f(\Lambda)$ by taking

$$P_k(\Lambda) = \bigoplus_{L_0<L_1<\cdots<L_k} \langle L_0, L_1, \cdots, L_k\rangle \prod_{i=0}^{k} e_{L_i} fR.$$

$$d_k\langle L_0, L_1, \cdots, L_k\rangle = \sum_{i=0}^{k} (-1)^i \langle L_0, \cdots, L_{i-1}, L_{i+1}, \cdots, L_k\rangle \prod_{j=0}^{k} e_{L_j} f.$$

$$d_0\langle L_0\rangle = e_{L_0} f.$$

A check that this creates an acyclic complex uses the easily checked result that, for all $p \in P_{k-1}(\Lambda)$ and for any M greater than all L's appearing in tuples of p,

(*) $$d_k(p^*M) = (-1)^k e_M p + (d_{k-1}p)^*M$$

where $^*M$ denotes the homomorphism induced on sums of smaller entries by

$$\langle L_0, \cdots, L_{k-1}\rangle^*M = \langle L_0, \cdots, L_{k-1}, M\rangle e_M.$$

Since p.d.$(I_f(\Lambda))=n$, $d_nP_n(\Lambda)$ is projective. Then there exists a directed subset $\Delta\subseteq\Lambda$ such that $\#\Delta=\aleph_n$ and $d_nP_n(\Delta)$ is a direct summand of $d_nP_n(\Lambda)$. There is a very slight change in the proof of 2.49 of **[11]** involved here—rephrase as in the proof of 2.53 of that paper to get the directed property. Now look at the subfield L' of K generated by all "second coordinates" of $\{f(1-e_J)|J\in\Delta\}$. It is $\aleph_n$-generated and so has at most $\aleph_n$ finite subfields. Hence there exists $M\in\Lambda$, $M>J$ for all $J\in\Delta$, such that M is not contained in L'. Let α generate M over F (every finite dimensional separable extension is generated by a single element). Then $\bar{\alpha}f(1-e_J)\neq 0$ by Remark 6, and since $\bar{\alpha}=\bar{\alpha}e_M$, $e_Mf(1-e_J)$ is nonzero for all $J\in\Delta$. Thus we may talk about the ideal $I_{e_Mf}(\Delta)$.

Using the relation (*) as in **[11]**, one sees that $e_Md_nP_n(\Delta)$ is actually a direct summand of $P_{n-1}(\Lambda)$ and hence of $e_MP_{n-1}(\Delta)$. Moreover, $\{e_MP_k(\Delta), d_k\}$ is a projective resolution of $I_{e_Mf}(\Delta)$ exactly as $\{P_k(\Lambda), d_k\}$ is one of $I_f(\Lambda)$. But $d_n(e_MP_n(\Delta))$ is a direct summand of $e_MP_{n-1}(\Delta)$, so p.d.$(I_{e_Mf}(\Delta))\leqq n-1$ and $\#\Delta=\aleph_n$, contradicting the induction hypothesis.

Putting Lemma 3, Corollary 8, and Proposition 9 together, we obtain

10. THEOREM. *If K is a separably generated extension field of F with separating transcendence basis B, such that $[K:F(B)]=\aleph_m$ then* bidim$(K)=\#B+m+1$.

PROOF. If K is separably algebraic, apply induction on m. For $m=-1$, this is the Noether-Hochschild result. If $m\geqq 0$,

$$\text{bidim}(K) = 1 + \text{p.d.}(I_1(\text{all finite extensions of } F \text{ in } K)) \geqq 1 + m$$

by 9 and $\leqq 1+m$ by 8. Lemma 3 finishes the proof.

3. **An upper bound in the locally separably generated case.** The reduction to the algebraic case to obtain a lower bound used in §§1 and 2 does not work in the case that K is only locally separably generated but not separably generated. Rosenberg and Zelinsky's example where $K=\bigcup_{n=0}^{\infty} F(x^{1/p^n})$, x transcendental and F of characteristic p, clearly illustrates this. Yet we can still get an upper bound as in the separably generated case by using Auslander's proposition in **[1]** to extend the argument for $\aleph_0$ in **[12]**.

11. LEMMA. *Let K' be a finitely generated, separably generated subfield of K. Then* $\text{p.d.}_{K'\otimes K'}(I_{K'})=\text{p.d.}_R(I_{K'}R)$.

PROOF. We use induction on transcendence degree $K'=n$. If $n=0$, $I_{K'}$ is generated by an idempotent, and so is $I_{K'}R$, so both are projective. If $n>0$, let x be an element of a separating transcendence basis for K'. Then $(\bar{x})$ is projective in $K'\otimes K'$ or $K\otimes K$ but not a direct summand, so for both rings, $\text{p.d.}(\tilde{I}_{K'}/(\bar{x}))=\text{p.d.}(\tilde{I}_{K'})$ (or $\text{p.d.}(\tilde{I}_{K'})+1$ if $n=1$), where $\tilde{I}_{K'}$ is the appropriate one of $I_{K'}$ or $I_{K'}R$. Since $R/(\bar{x})\simeq K\otimes_{F(x)} K$ and $I_{K'}/(\bar{x})$ is the kernel of the multiplication map from $K'\otimes_{F(x)} K'\to K'$, the result follows from the induction hypothesis and Lemma 1.

12. COROLLARY. *Let K be locally separably generated*, $\text{tr deg}(K)=n$, *K $\aleph_k$-generated. Then* $\text{bidim}(K)\leqq n+k+1$.

PROOF. The finitely generated separably generated subfields K' of K form a directed set under $\subseteq$, and there are $\aleph_k$ of them. Then $\{I_{K'}R\}$ is a directed system of submodules of I_K, each of projective dimension $\leqq n$. By **[11**, Proposition 2.43], $\text{p.d.}(\sum I_{K'}R)\leqq n+k+1$. Since K is locally separably generated, $\sum I_{K'}R=I_K$.

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