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A GERŠGORIN INCLUSION SET FOR THE FIELD OF VALUES OF A FINITE MATRIX

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ABSTRACT. An easily computed Geršgorin type inclusion set for the field of values of an n by n complex matrix is presented. Some functional properties of this inclusion set parallel those of the field of values, and illustrative examples are given.

1. **Introduction.** Let $M_n(C)$ denote the set of *n by n matrices* over the complex field. For $A=(a_{ij}) \in M_n(C)$, define

$$R_i(A) = \sum_{j=1, j\neq i}^{n} |a_{ij}| \quad \text{and} \quad C_j(A) = \sum_{i=1, i\neq j}^{n} |a_{ij}|$$

and let

$$G_r(A) = \bigcup_{i=1}^{n} \{z : |z - a_{ii}| \leqq R_i(A)\},$$

$$G_c(A) = \bigcup_{j=1}^{n} \{z : |z - a_{jj}| \leqq C_j(A)\}.$$

The well known theorem of Geršgorin [2] notes that the *spectrum* $\sigma(A)$ of A is contained in $G_r(A) \cap G_c(A)$.

Denote the *field of values* of $A \in M_n(C)$ by

$$F(A) = \{xAx^* : x \in C^n, xx^* = 1\}.$$

The bounded complex set $F(A)$ is convex, $\sigma(A) \subseteq F(A)$, and $F(A)$ is invariant under unitary similarities of A. In case A is normal, $F(A)$ is the convex hull of $\sigma(A)$.

For a set S in the complex plane, let $\mathrm{Co}(S)$ be its *convex hull*, let $g_i(A)=(R_i(A)+C_i(A))/2$ and define

$$G(A) = \mathrm{Co}\left(\bigcup_{i=1}^{n} \{z : |z - a_{ii}| \leqq g_i(A)\}\right).$$

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It is the goal of this note to present for $F(A)$ an analog of Geršgorin's theorem: $F(A)\subseteq G(A)$. Considered as set valued functions of a matrix argument G and F also share several functional properties.

By the *sum of two sets* S_1+S_2 we shall mean $\{x_1+x_2:x_1 \in S_1, x_2 \in S_2\}$ and let $R=\{z:\mathrm{Re}(z)>0\}$ denote the *right complex half-plane*.

2. **Functional properties of G and F.** G and F may be considered as functions from $M_n(C)$ into the class of convex subsets of the complex plane. As such they have many functional properties in common. We have already noted that $F(A)$ and $G(A)$ both contain $\sigma(A)$ and the first four of the following remarks note functional properties of G which are well known for F.

REMARK 1. For any complex number α, $G(A-\alpha I)=G(A)+\{-\alpha\}$ and $G(\alpha A)=\alpha G(A)$.

REMARK 2. If $A_0 \in M_k(C)$ is a principal submatrix of $A \in M_n(C)$, $k\leqq n$, then $G(A_0)\subseteq G(A)$.

PROOF. This follows from the observation that if A_0 is determined by the indices $i_1, \cdots, i_k$, then $g_j(A_0)\leqq g_{i_j}(A), j=1, \cdots, k$.

REMARK 3. G is subadditive. That is for A, $B \in M_n(C)$, $G(A+B)\subseteq G(A)+G(B)$.

PROOF. Let $A=(a_{ij})$, $B=(b_{ij})$. Because of the triangle $g_i(A+B)\leqq g_i(A)+g_i(B)$ for all $i=1, \cdots, n$. It follows that

$$\{z:|z-(a_{ii}+b_{ii})| \leqq g_i(A+B)\} \subseteq \{z:|z-a_{ii}| \leqq g_i(A)\}+\{z:|z-b_{ii}| \leqq g_i(B)\}$$

which implies $G(A+B)\subseteq G(A)+G(B)$.

REMARK 4. If $A \in M_n(C)$ is diagonal, then $F(A)=G(A)$.

REMARK 5. Unlike F, $G(A)$ is not invariant under unitary similarities of A.

3. **Main result.** We next show that $G(A)$ is also an upper estimate for $F(A)$ for all $A \in M_n(C)$.

LEMMA 1. *If* $G(A)\subseteq R$, *then* $F(A)\subseteq R$.

PROOF. Let $A=(a_{ij})$; let the Hermitian part of A be $(b_{ij})=B=(A+A^*)/2$; and suppose $G(A)\subseteq R$ which means $\mathrm{Re}(a_{ii})>g_i(A)$. Since $R_i(A^*)=C_i(A)$ and because of the triangle inequality, $R_i(B)\leqq g_i(A)$ and we have $b_{ii}=\mathrm{Re}(a_{ii})>g_i(A)\geqq R_i(B)$. Then since $\sigma(B)\subseteq G_r(B)$ by Geršgorin's theorem and since $G_r(B)\subseteq R$ because $b_{ii}>R_i(B)$ we obtain that $\sigma(B)\subseteq R$. But since B is Hermitian $F(B)=\mathrm{Co}(\sigma(B))$ and thus $F(B)\subseteq R$. Now $A=B+C$ where $C=(A-A^*)/2$. Since $F(C)$ is pure imaginary $F(B)+F(C)\subseteq R$ and by the subadditivity of F, $F(A)\subseteq R$ as was to be shown.

LEMMA 2. *If* $0 \notin G(A)$, *then* $0 \notin F(A)$.

PROOF. Suppose $0 \notin G(A)$. Since $G(A)$ is convex, there is a θ, $0 \leqq \theta < 2\pi$, such that $G(e^{i\theta}A) = e^{i\theta}G(A) \subseteq R$. By Lemma 1 this implies $F(e^{i\theta}A) \subseteq R$, and since $F(A) = e^{-i\theta}F(e^{i\theta}A)$, it follows that $0 \notin F(A)$.

THEOREM. *For all* $A \in M_n(C)$, $F(A) \subseteq G(A)$.

PROOF. Suppose $\alpha \in F(A)$. Then $0 \in F(A - \alpha I)$ and by the contrapositive of Lemma 2, $0 \in G(A - \alpha I)$. Because of Remark 1, we may conclude that $\alpha \in G(A)$. Thus $F(A) \subseteq G(A)$ which completes the proof.

If we denote the *numerical radius*, $\max_{\alpha \in F(A)} |\alpha|$, of $A \in M_n(C)$ by $r(A)$, then we may obtain an estimate for $r(A)$ by the preceding theorem.

COROLLARY. *For* $A \in M_n(C)$,

$$r(A) \leqq \max_i(|a_{ii}| + g_i(A)) = \max_i \left(\sum_{j=1}^{n} \frac{|a_{ij}| + |a_{ji}|}{2} \right).$$

PROOF. Because of the theorem, it merely suffices to note that $\max_{\alpha \in G(A)} |\alpha| = \max_i(|a_{ii}| + g_i(A))$ which is valid since $G(A)$ is the convex hull of a closed set whose largest element in absolute value is

$$\max_i(|a_{ii}| + g_i(A)).$$

4. **Examples and further remarks.** Our first example shows that $G(A)$ gives the most economical general estimate of its type for $F(A)$, and the third shows how much of an overestimate $G(A)$ can be in an extreme case. We then give an application which is a sufficient condition for $F(A)$ to be a circle (with interior).

EXAMPLE 1. Let $A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$. It then may be computed that $F(A) = G(A)$ which is the unit circle.

Suppose $s(x, y)$ is a function of the two nonnegative real variables x and y, and let $A = (a_{ij})$ be an arbitrary element of $M_n(C)$. Let $s_i = s(R_i(A), C_i(A))$ and define

$$G_s(A) = \mathrm{Co}\left(\bigcup_{i=1}^{n} \{z : |z - a_{ii}| \leqq s_i\} \right).$$

If $F(A) \subseteq G_s(A)$ for all A, it then follows from Example 1 that $G(A) \subseteq G_s(A)$ and $s(x, y) \geqq (x + y)/2$, and $G(A)$ is the best upper estimate of this type.

EXAMPLE 2. That the geometric mean does not provide an upper estimate for $F(A)$ is shown by letting $A = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$ and $s(x, y) = (xy)^{1/2}$. Then $G_s(A)$ is the circle about 2 of radius $3^{1/2}$, but it is easily seen that $0 \in F(A)$ so that $F(A) \nsubseteq G_s(A)$.

EXAMPLE 3. Let

$$A = \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ \cdot & \cdot & & \\ \cdot & \cdot & 0 & \\ \cdot & \cdot & & \\ 1 & 0 & & \end{bmatrix} \in M_n(C).$$

Then $F(A)$ may be computed to be $[-(n-1)^{1/2}, (n-1)^{1/2}]$ with $r(A)=(n-1)^{1/2}$. Since $G(A)$ is the circle about the origin of radius $n-1$, $G(A)$ is a heavy overestimate in this extreme case. However, if

$$B = \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \\ \cdot & \cdot & & \\ \cdot & 1 & \cdot & \\ \cdot & & \cdot & \\ 1 & & & 0 \end{bmatrix} \in M_n(C),$$

$G(B)$ is again the circle of radius $n-1$ about the origin. But $F(B)=[1-n, n-1]$ with $r(B)=n-1$.

REMARK 6. It is easy to see [**1**] that any $A \in M_n(C)$ may be unitarily transformed to $(b_{ij})=B=U^*AU$ with $b_{ii}=T_r(A)/n$, $i=1, \cdots, n$. Suppose without loss of generality that $g_1(B) \geqq g_i(B)$, $i=2, \cdots, n$, also. Then if row 1 and column 1 of B contain at most one nonzero entry besides b_{11} (for instance b_{1n}), $F(A)$ is a circle about $T_r(A)/n$ (of radius $g_1(B)=|b_{1n}|/2$). Let $a=T_r(A)/n$ and this follows since

$$G\left(\begin{bmatrix} a & b_{1n} \\ 0 & a \end{bmatrix}\right) = F\left(\begin{bmatrix} a & b_{1n} \\ 0 & a \end{bmatrix}\right) \subseteq F(B) \subseteq \{z : |z-a| \leqq g_1(B)\}$$
$$= G\left(\begin{bmatrix} a & b_{1n} \\ 0 & a \end{bmatrix}\right)$$

and $F(A)=F(B)$.

REFERENCES

1. W. V. Parker, *Sets of complex numbers associated with a matrix*, Duke Math. J. **15** (1948), 711–715. MR **10**, 230.

2. S. Geršgorin, *Über die Abgrenzung der Eigenwerte einer Matrix*, Izv. Akad. Nauk SSSR **7** (1931), 749–754.

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