A GERSGORIN INCLUSION SET FOR THE FIELD OF VALUES OF A FINITE MATRIX

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ABSTRACT. An easily computed Gersgorin type inclusion set for the field of values of an $n$ by $n$ complex matrix is presented. Some functional properties of this inclusion set parallel those of the field of values, and illustrative examples are given.

1. Introduction. Let $M_n(\mathbb{C})$ denote the set of $n$ by $n$ matrices over the complex field. For $A = (a_{ij}) \in M_n(\mathbb{C})$, define

$$R_i(A) = \sum_{j=1, j \neq i}^{n} |a_{ij}|$$
$$C_j(A) = \sum_{i=1, i \neq j}^{n} |a_{ij}|$$

and let

$$G_r(A) = \bigcup_{i=1}^{n} \{ z : |z - a_{ii}| \leq R_i(A) \}$$
$$G_c(A) = \bigcup_{j=1}^{n} \{ z : |z - a_{jj}| \leq C_j(A) \}.$$ 

The well known theorem of Gersgorin [2] notes that the spectrum $\sigma(A)$ of $A$ is contained in $G_r(A) \cap G_c(A)$.

Denote the field of values of $A \in M_n(\mathbb{C})$ by

$$F(A) = \{ xAx^* : x \in \mathbb{C}^n, xx^* = 1 \}.$$ 

The bounded complex set $F(A)$ is convex, $\sigma(A) \subseteq F(A)$, and $F(A)$ is invariant under unitary similarities of $A$. In case $A$ is normal, $F(A)$ is the convex hull of $\sigma(A)$.

For a set $S$ in the complex plane, let $Co(S)$ be its convex hull, let $g_i(A) = (R_i(A) + C_i(A))/2$ and define

$$G(A) = Co \left( \bigcup_{i=1}^{n} \{ z : |z - a_{ii}| \leq g_i(A) \} \right).$$
It is the goal of this note to present for $F(A)$ an analog of Gersgorin’s theorem: $F(A) \subseteq G(A)$. Considered as set valued functions of a matrix argument $G$ and $F$ also share several functional properties.

By the sum of two sets $S_1 + S_2$ we shall mean \{\(x_1 + x_2: x_1 \in S_1, x_2 \in S_2\)\} and let $R = \{z: \text{Re}(z) > 0\}$ denote the right complex half-plane.

2. Functional properties of $G$ and $F$. $G$ and $F$ may be considered as functions from $M_n(C)$ into the class of convex subsets of the complex plane. As such they have many functional properties in common. We have already noted that $F(A)$ and $G(A)$ both contain $\sigma(A)$ and the first four of the following remarks note functional properties of $G$ which are well known for $F$.

**Remark 1.** For any complex number $\alpha$, $G(A - \alpha I) = G(A) + \{-\alpha\}$ and $G(\alpha A) = \alpha G(A)$.

**Remark 2.** If $A_0 \in M_k(C)$ is a principal submatrix of $A \in M_n(C)$, $k \leq n$, then $G(A_0) \subseteq G(A)$.

**Proof.** This follows from the observation that if $A_0$ is determined by the indices $i_1, \cdots, i_k$, then $g_{j}(A_0) \leq g_{i_j}(A)$, $j = 1, \cdots, k$.

**Remark 3.** $G$ is subadditive. That is for $A, B \in M_n(C)$, $G(A + B) \subseteq G(A) + G(B)$.

**Proof.** Let $A = (a_{ij}), B = (b_{ij})$. Because of the triangle inequality $g_i(A + B) \leq g_i(A) + g_i(B)$ for all $i = 1, \cdots, n$. It follows that

\[
\{z: |z - (a_{ii} + b_{ii})| \leq g_i(A + B)\} \\
\subseteq \{z: |z - a_{ii}| \leq g_i(A)\} + \{z: |z - b_{ii}| \leq g_i(B)\}
\]

which implies $G(A + B) \subseteq G(A) + G(B)$.

**Remark 4.** If $A \in M_n(C)$ is diagonal, then $F(A) = G(A)$.

**Remark 5.** Unlike $F$, $G(A)$ is not invariant under unitary similarities of $A$.

3. Main result. We next show that $G(A)$ is also an upper estimate for $F(A)$ for all $A \in M_n(C)$.

**Lemma 1.** If $G(A) \subseteq R$, then $F(A) \subseteq R$.

**Proof.** Let $A = (a_{ij})$; let the Hermitian part of $A$ be $(b_{ij}) = B = (A + A^*)/2$; and suppose $G(A) \subseteq R$ which means $\text{Re}(a_{ii}) > g_i(A)$. Since $R_i(A^*) = C_i(A)$ and because of the triangle inequality, $R_i(B) \leq g_i(A)$ and we have $b_{ii} = \text{Re}(a_{ii}) > g_i(A) \geq R_i(B)$. Then since $\sigma(B) \leq G_i(B)$ by Gersgorin’s theorem and since $G_i(B) \subseteq R$ because $b_{ii} > R_i(B)$ we obtain that $\sigma(B) \subseteq R$. But since $B$ is Hermitian $F(B) = C_0(\sigma(B))$ and thus $F(B) \subseteq R$. Now $A = B + C$ where $C = (A - A^*)/2$. Since $F(C)$ is pure imaginary $F(B) + F(C) \subseteq R$ and by the subadditivity of $F$, $F(A) \subseteq R$ as was to be shown.
Lemma 2. If $0 \notin G(A)$, then $0 \notin F(A)$.

Proof. Suppose $0 \notin G(A)$. Since $G(A)$ is convex, there is a $\theta$, $0 \leq \theta < 2\pi$, such that $G(e^{i\theta}A) = e^{i\theta}G(A) \subseteq R$. By Lemma 1 this implies $F(e^{i\theta}A) \subseteq R$, and since $F(A) = e^{-i\theta}F(e^{i\theta}A)$, it follows that $0 \notin F(A)$.

Theorem. For all $A \in M_n(C)$, $F(A) \subseteq G(A)$.

Proof. Suppose $\alpha \in F(A)$. Then $0 \in F(A - \alpha I)$ and by the contrapositive of Lemma 2, $0 \in G(A - \alpha I)$. Because of Remark 1, we may conclude that $\alpha \in G(A)$. Thus $F(A) \subseteq G(A)$ which completes the proof.

If we denote the numerical radius, $\max_{a \in F(A)} |a|$, of $A \in M_n(C)$ by $r(A)$, then we may obtain an estimate for $r(A)$ by the preceding theorem.

Corollary. For $A \in M_n(C)$,

$$r(A) \leq \max_i (|a_{ii}| + g_i(A)) = \max_i \left( \sum_{j=1}^n |a_{ij}| + |a_{ji}| \right).$$

Proof. Because of the theorem, it merely suffices to note that $\max_{a \in G(A)} |a| = \max_i (|a_{ii}| + g_i(A))$ which is valid since $G(A)$ is the convex hull of a closed set whose largest element in absolute value is

$$\max_i (|a_{ii}| + g_i(A)).$$

4. Examples and further remarks. Our first example shows that $G(A)$ gives the most economical general estimate of its type for $F(A)$, and the third shows how much of an overestimate $G(A)$ can be in an extreme case. We then give an application which is a sufficient condition for $F(A)$ to be a circle (with interior).

Example 1. Let $A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$. It then may be computed that $F(A) = G(A)$ which is the unit circle.

Suppose $s(x, y)$ is a function of the two nonnegative real variables $x$ and $y$, and let $A = (a_{ij})$ be an arbitrary element of $M_n(C)$. Let $s = s(R_i(A), C_i(A))$ and define

$$G_s(A) = Co\left( \bigcup_{i=1}^n \{ z : |z - a_{ii}| \leq s_i \} \right).$$

If $F(A) \subseteq G_s(A)$ for all $A$, it then follows from Example 1 that $G(A) \subseteq G_s(A)$ and $s(x, y) \geq (x+y)/2$, and $G(A)$ is the best upper estimate of this type.

Example 2. That the geometric mean does not provide an upper estimate for $F(A)$ is shown by letting $A = \begin{pmatrix} 1 & 3 \\ 2 & 0 \end{pmatrix}$ and $s(x, y) = (xy)^{1/2}$. Then $G_s(A)$ is the circle about 2 of radius $3^{1/2}$, but it is easily seen that $0 \notin F(A)$ so that $F(A) \nsubseteq G_s(A)$. 

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Example 3. Let

\[ A = \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix} \in M_n(C). \]

Then \( F(A) \) may be computed to be \([-(n-1)^{1/2}, (n-1)^{1/2}]\) with \( r(A) = (n-1)^{1/2} \). Since \( G(A) \) is the circle about the origin of radius \( n-1 \), \( G(A) \) is a heavy overestimate in this extreme case. However, if

\[ B = \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix} \in M_n(C), \]

\( G(B) \) is again the circle of radius \( n-1 \) about the origin. But \( F(B) = [1-n, n-1] \) with \( r(B) = n-1 \).

Remark 6. It is easy to see [1] that any \( A \in M_n(C) \) may be unitarily transformed to \( (b_{ij}) = B = U^*AU \) with \( b_{ii} = T_r(A)/n, i = 1, \ldots, n \). Suppose without loss of generality that \( g_1(B) \geq g_i(B), i = 2, \ldots, n \), also. Then if row 1 and column 1 of \( B \) contain at most one nonzero entry besides \( b_{11} \) (for instance \( b_{1n} \)), \( F(A) \) is a circle about \( T_r(A)/n \) (of radius \( g_1(B) = |b_{1n}|/2 \)). Let \( a = T_r(A)/n \) and this follows since

\[
G\left(\begin{bmatrix} a & b_{1n} \\ 0 & a \end{bmatrix}\right) = F\left(\begin{bmatrix} a & b_{1n} \\ 0 & a \end{bmatrix}\right) \subseteq F(B) \subseteq \{z: |z - a| \leq g_1(B)\}
\]

and \( F(A) = F(B) \).

References


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