FINITE DIMENSIONAL GROUP RINGS

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Abstract. A ring is right finite dimensional if it contains no infinite direct sum of right ideals. We prove that if a group ring $RG$ is right finite dimensional if and only if the group ring $RG$ is right finite dimensional. A ring $R$ is a self-injective cogenerator ring if $R$ is injective and $R$ is a cogenerator in the category of unital right $R$-modules; this means that each right unital $R$-module can be embedded in a direct product of copies of $R$. Let $G$ be a finite group where the order of $G$ is a unit in $R$. Then the group ring $RG$ is a self-injective cogenerator ring if and only if $R$ is a self-injective cogenerator ring. Additional applications are given.

1. Introduction. Let $R$ always denote an associative ring with 1 and $G$ a group with order $|G|$. The group ring of a group $G$ and a ring $R$ is the ring of all formal sums $\sum_{g \in G} r(g)g$ with $r(g) \in R$ and with only finitely many nonzero $r(g)$ [7]. For a right finite dimensional ring $R$, there exists an integer $n$ such that $R$ contains a direct sum of $n$-summands and the number of summands of any other direct sum in $R$ is at most $n$. In this case, we write $\text{dim } R = n$. The ring $R$ will be considered as a right $R$-module $R_R$ and by finite dimensional we shall mean right finite dimensional.

It is known that if $H$ is any semigroup with 1, then $RH$ is a ring. In particular, the polynomial ring is a special case of this construction. Shock has shown that the right finite dimensional property carries over to polynomial rings [10]. This paper extends this result to group rings.

If $R$ is a subring of $Q$ and the identity of $R$ is also the identity of $Q$, then $R$ is a right order in $Q$ if

(a) every nonzero divisor of $R$ is a unit in $Q$, and

(b) every element of $Q$ can be written in the form of $cd^{-1}$ where $c$ and $d$ are in $R$ and $d$ is a nonzero divisor of $R$. We prove that if $G$ is a finite group, then $R$ is a right order in a self-injective cogenerator ring and the order of no finite normal subgroup of $G$ is a zero-divisor in $R$.

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if and only if $RG$ is a right order in a self-injective cogenerator ring. Let $G$ be a free abelian group. If $R$ is a right order in a right Artinian ring then $RG$ is a right order in a right Artinian ring.

2. Finite dimensional group rings. It is always true that if $RG$ is finite dimensional then $R$ is finite dimensional; however, the converse is not in general true.

Example 2.1. There exists a finite dimensional ring $R$ and a group $G$ such that the group ring $RG$ is not finite dimensional. Let $R$ be a field of characteristic zero and $G = \bigoplus \sum_{C_p}$ (for all prime $p$), where $C_p$ is a cyclic group of order $p$. Then $RG$ is not finite dimensional. This follows from the fact that $RG$ is regular and the right ideal $\omega(C_p)$ of $RG$ generated by $\{1-h|h \in C_p\}$ is principal [2]. So the question naturally arises as to when the group ring $RG$ is finite dimensional.

Proposition 2.2 (Shok [10]). A ring $R$ is finite dimensional if and only if the polynomial ring $R[x_1, x_2, \ldots]$ is finite dimensional. Furthermore, $\dim R = \dim R[x_1, x_2, \ldots]$.

Proof. See Theorem 2.6 of [10].

Let $R$ be a subring of $S$, then we call $S$ a ring of right quotients of $R$, if for every $0 \neq s \in S$ and for every $s' \in S$, there exists $r \in R$ such that $sr \neq 0$ and $s'r \in R$. Let $Q(R)$ denote the complete ring of quotients of $R$. It is well known that $R$ is finite dimensional if and only if $Q(R)$ is, and in this case $\dim R = \dim Q(R)$. It is also known that if $S$ is a ring of right quotients of $R$ then $Q(R)$ is the complete ring of quotients of $S$ [4].

Theorem 2.3. Let $G$ be an infinite cyclic group, then $R$ is finite dimensional if and only if $RG$ is finite dimensional. Furthermore, $\dim R = \dim RG$.

Proof. Let $S$ be a multiplicative semigroup isomorphic to the nonnegative integers. Then $S$ is a semigroup with identity and is generated by the nonnegative powers of some element, say $g$. By Proposition 2.2, it is clear that $RS$ is finite dimensional, since $RS$ is just a polynomial ring in the variable $g$. Now $S$ can be embedded in an infinite cyclic group $G$, which is generated by all powers of $g$. We need only show that $RG$ is a ring of right quotients of $RS$. Let $r_1, r_2 \in RG$ with

$$0 \neq r_1 = r_1(g_1)g_1 + \cdots + r_1(g_n)g_n = r_1(g_1)g^{a_1} + \cdots + r_1(g_n)g^{a_n}$$

and

$$r_2 = r_2(h_1)h_1 + \cdots + r_2(h_m)h_m = r_2(h_1)g^{b_1} + \cdots + r_2(h_m)g^{b_m}.$$
Let \( k = \max\{|a_i|, |b_j|\} \) for all \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \). It is clear that 
\[ r = g^k \in RS, \ r_1r \neq 0, \text{ and } r_2r \in RS. \] Hence, \( RG \) is finite dimensional. Also, 
\[ \dim Q(RS) = \dim RS = \dim R \] shows that \( \dim R = \dim RG \). The converse is clear.

A free abelian group is a group which is a direct sum of infinite cyclic groups.

**Corollary 2.4.** Let \( G \) be a free abelian group, then \( R \) is finite dimensional if and only if \( RG \) is finite dimensional. Furthermore, \( \dim R = \dim RG \).

**Proof.** Let \( H = S_1 \oplus S_2 \oplus \cdots \) where each \( S_i \) is a multiplicative semigroup isomorphic to the nonnegative integers. If \( R \) is finite dimensional then \( RH \) is finite dimensional by Proposition 2.2. Let \( G = G_1 \oplus G_2 \oplus \cdots \), where \( S_i \) is embedded in the infinite cyclic group \( G_i \), and now show that \( RG \) is a ring of right quotients of \( RH \). The details are omitted. The converse and \( \dim R = \dim RG \) follow easily.

**Lemma 2.5.** For a finite group \( G \), the group ring \( RG \) is finite dimensional if and only if the ring \( R \) is finite dimensional. Also, \( \dim R \leq \dim RG \leq \dim R \cdot |G| \).

**Proof.** Let \( G \) be finite, then \( RG_R \) is \( R \)-isomorphic to a direct sum of \( |G| \) copies of the finite dimensional \( R \)-module \( R \). Hence, \( RG \) is a finite dimensional \( R \)-module and therefore a finite dimensional \( RG \)-module. The converse and inequalities are clear.

**Theorem 2.6.** Let \( G \) be a finitely generated abelian group, then \( R \) is finite dimensional if and only if \( RG \) is finite dimensional. If \( H \) is the torsion subgroup of \( G \), then \( \dim R \leq \dim RG \leq \dim R \cdot |H| \).

**Proof.** If \( G \) is a finitely generated abelian group then \( G = G_1 \oplus G_2 \oplus \cdots \oplus G_n \oplus H \) where \( |H| < \infty \) and \( G_i \) for \( 1 \leq i \leq n \) is an infinite cyclic group. As in [2, p. 673], we define \( A_1 = RG_1, \ A_2 = A_1G_2, \cdots, \ A_n = A_{n-1}G_n, \) and \( A = A_nH \); clearly \( RG \cong A \). By Corollary 2.4 and Lemma 2.5, we see by induction that \( A \) is finite dimensional and consequently \( RG \) is finite dimensional. The converse and inequalities follow easily.

3. **Applications.** Let \( Z(R) \) denote the right singular ideal of \( R \) (4).

**Lemma 3.1.** Let \( G \) be a free abelian group, then \( Z(RG) = Z(R)G \).

**Proof.** The proof uses the same technique as the proof of Theorem 2.7 of [10].

**Proposition 3.2 (Connell, [2]).** The group ring \( RG \) is semiprime if and only if \( R \) is semiprime and the order of no finite normal subgroup is a zero-divisor in \( R \).

**Proof.** See the appendix of [4].
It is well known that a semiprime Goldie ring is a semiprime, finite dimensional ring with zero singular ideal.

**Corollary 3.3.** Let $G$ be a free abelian group. A ring $R$ is a semiprime Goldie ring if and only if $RG$ is a semiprime Goldie ring.

**Proof.** The proof is immediate.

**Proposition 3.4 (Burgess, [1]).** If $Z(RG)=0$, then $Z(R)=0$ and the order of every finite normal subgroup of $G$ is a nonzero-divisor in $R$.

**Proof.** See Theorem 4.8 of [1].

A *locally normal group* is one in which every finite subset is contained in a finite normal subgroup.

**Proposition 3.5 (Burgess, [1]).** Assume that $G$ is locally normal and the order of every finite normal subgroup of $G$ is a nonzero-divisor in $R$. If $Z(R)=0$, then $Z(RG)=0$.

**Proof.** See 4.9 of [1].

**Corollary 3.6.** Let $G$ be a finitely generated abelian group. Then $R$ is a semiprime Goldie ring and the order of every finite normal subgroup of $G$ is a nonzero-divisor in $R$ if and only if $RG$ is a semiprime Goldie ring.

**Proof.** The proof is immediate using the construction in the proof of Theorem 2.6.

A right ideal of a ring $R$ is said to be *essential* if it has nonzero intersection with every nonzero right ideal of $R$. A right ideal $D$ of $R$ is *dense* if for every $0 \neq r_1 \in R$ and for every $r_2 \in R$ there exists $r \in R$ such that $r_1 r \neq 0$ and $r_2 r \in D$. We denote the Jacobson radical of $R$ by Rad $R$. A right ideal $A$ is said to be *small* if for every right ideal $B$, $A+B=R$ implies $B=R$. It is known that $A$ is small if and only if $A \subseteq \text{Rad } R$.

The following remarks are well known.

**Remark 3.7.** A right ideal $D$ is dense in $R$ if and only if $DG$ is dense in $RG$.

**Remark 3.8.** A right ideal $L$ is essential in $R$ if and only if $LG$ is essential in $RG$.

A right ideal $B$ is *rationally closed* in $R$ if $x^{-1}B = \{ r \in R \mid xr \in B \}$ is not dense for all $x \in R - B$. Let $I(R)$ denote the injective hull of $R$, then $B$ is rationally closed in $R$ if there exists a subset $S$ of $I(R)$ such that $B = \{ x \in R \mid Sx = 0 \}$ [8].

**Lemma 3.9.** A right ideal $K$ of $R$ is rationally closed in $R$ if and only if $KG$ is rationally closed in $RG$. 
Proof. If $K$ is rationally closed then there exists a subset $S \subseteq I(R)$ such that $K = \{x \in R|Sx = 0\}$. We will show that $KG = \{x \in RG|SGx = 0\}$. Let $x \in KG$ then $SGx = 0$ since $Sk = 0$ for all $k \in K$. Hence $x \in \{x \in RG|SGx = 0\}$. Now suppose $0 \neq x \notin KG$. We want to show there exists $y \in SG$ such that $yx \neq 0$. Let $x = r_1(g_1)g_1 + \cdots + r_n(g_n)g_n$, since $x \notin KG$ there exists $r_i(g_i)$ such that $r_i(g_i) \notin K$. $K$ is rationally closed so there exists $0 \neq s \in S$ such that $sr_i(g_i) \neq 0$. Hence, $sx \neq 0$ implies $x \notin \{x \in RG|SGx = 0\}$.

Conversely, suppose $K$ is not rationally closed in $R$, then there exists $x \in R - K$ such that $x^{-1}K$ is dense in $R$. Thus $(x^{-1}K)G = x^{-1}KG$ is dense in $RG$ and hence $KG$ is not rationally closed in $RG$.

Proposition 3.10 (Renault, [6]). The group ring $RG$ is self-injective if and only if $R$ is self-injective and $G$ is finite.

Proof. See [6].

Lemma 3.11 (Shock, [9]). Let $R$ be a self-injective ring. Then $R$ is a cogenerator if and only if $R$ is right finite dimensional and $Z(R)$ is rationally closed.

Proof. See Proposition 2 of [9].

If $R$ is a self-injective ring then $Z(R) = \text{Rad } R$ [4]. It is known that if $R$ is self-injective and finite dimensional then $R/\text{Rad } R$ is completely reducible.

Theorem 3.12. Let $G$ be a finite group where the order of $G$ is a unit in $R$, then $R$ is a self-injective cogenerator ring if and only if $RG$ is a self-injective cogenerator ring.

Proof. Let $R$ be a self-injective cogenerator ring. It is clear that $RG$ is finite dimensional and injective. By Lemma 3.11, we need only show that $Z(RG)$ is rationally closed. It is clear that if $R$ contains no proper dense right ideals then every right ideal is rationally closed and conversely. So, we shall show that $RG$ contains no proper dense right ideals. Let $D$ be a dense right ideal of $RG$. Then $D + Z(R)G$ is dense and by Proposition 5.1 of [8], $(D + Z(R)G)/Z(R)G$ is dense in $RG/Z(R)G$ since $Z(R)G$ is rationally closed. Clearly, $RG/Z(R)G$ and $R/Z(R)$ are completely reducible. Therefore, $(R/Z(R))G \cong RG/Z(R)G$ is completely reducible [2] and thus $RG/Z(R)G$ contains no proper dense right ideals. Hence, $D + Z(R)G = RG$. But $Z(R)G \subset Z(RG) = \text{Rad } RG$ implies $Z(R)G$ is small. Hence, $D = RG$.

Conversely, let $D$ be dense in $R$, $D \neq R$, then $DG$ is dense in $RG$ and $DG \neq RG$. 
**Lemma 3.13** (Shock, [9]). Suppose that $Z(Q(R))$ is the Jacobson radical of $Q(R)$ and is rationally closed. If $Q(R)/Z(Q(R))$ is a completely reducible ring and $R/Z(R)$ is semiprime, then $R$ is a right order in $Q(R)$.

**Proof.** See Proposition 4 of [9].

**Theorem 3.14.** Let $G$ be a finite group, then $R$ is a right order in a self-injective cogenerator ring and the order of no finite normal subgroup of $G$ is a zero-divisor in $R$ if and only if $RG$ is a right order in a self-injective cogenerator ring.

**Proof.** Let $R$ be a right order in a self-injective cogenerator ring $Q$, then $Q = Q(R)$. By 3.6 of [1], we have $Q(RG) \cong Q(R)G$ and thus by Theorem 3.12 $Q(RG)$ is a self-injective cogenerator ring. It is now clear that both $Q(RG)/Z(Q(RG))$ and $Q(R)/Z(Q(R))$ are completely reducible. Also, it is clear that $Q(RG)/Z(Q(RG))G$ is completely reducible and that $RG/Z(RG)$ is semiprime. By Lemma 3.13 we need only to show that $RG/Z(RG)$ is semiprime. To do this, we first show that $Z(RG) = Z(RG)$. It is sufficient to show that $Z(Q(RG)) = Z(Q(R))G$ since $Z(RG) = Z(Q(RG)) \cap RG = Z(Q(RG)) \cap RG = Z(Q(R))G \cap RG = Z(RG)$. Now $(Q(R)/(Z(Q(R))))G \cong Q(RG)/Z(Q(R))G \cong Q(R)/Z(Q(R))G$. Hence, $Z(Q(R))G = Z(Q(RG))$ since $Q(RG)/Z(Q(R))G$ is completely reducible. The converse follows similarly.

In [12] Smith showed that if $G$ is a poly- (cyclic or finite) group and $R$ is a right order in a right Artinian ring then $RG$ is a right order in a right Artinian ring. We extend this result to a class of group rings, where $G$ need not be poly- (cyclic or finite), using a method of Small [11].

**Theorem 3.15.** Let $G$ be a free abelian group. If $R$ is a right order in a right Artinian ring then $RG$ is a right order in a right Artinian ring.

**Proof.** It is clear that $rad (RG) = (rad R)G$ when $G$ is free abelian. We now use the same argument as in Theorem 3.6 of [10].

**References**


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