LIE AND JORDAN STRUCTURE IN PRIME RINGS WITH DERIVATIONS

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Abstract. In this paper Lie ideals and Jordan ideals of a prime ring \( R \) together with derivations on \( R \) are studied. The following results are proved: Let \( R \) be a prime ring, \( U \) be a Lie ideal or a Jordan ideal of \( R \) and \( d \) be a nonzero derivation of \( R \) such that
\[
ud(u) - d(u)u \text{ is central in } R \quad \text{for all } u \in U.
\]
(i) If the characteristic of \( R \) is different from 2 and 3, then \( U \) is central in \( R \). (ii) If \( R \) has characteristic 3 and \( U \) is a Jordan ideal then \( U \) is central in \( R \); further, if \( U \) is a Lie ideal with \( u^2 \in U \) for all \( u \in U \), then \( U \) is central in \( R \). The case when \( R \) has characteristic 2 is also studied. These results extend some due to Posner [2].

1. Introduction. A theorem of Posner [2] states that if \( R \) is a prime ring, and \( d \) is a nonzero derivation of \( R \) such that, for all \( r \in R \),
\[
rd(r) - d(r)r \text{ is in the centre of } R,
\]
then \( R \) is commutative. Our object is to generalize this theorem to Lie and Jordan ideals of \( R \).

All rings considered here are associative. Let \( R \) be a ring and \( Z \) be its centre. For \( x, y \in R \), \([x, y] = xy - yx\). For \( a \in R \), let \( I_a \) denote the inner derivation of \( R \) by \( a \); i.e., \( I_a(x) = ax - xa \) for all \( x \in R \). Throughout the paper \( d \) denotes a nonzero derivation of \( R \). For definitions see [1].

2. Basic lemmas. We begin with some preliminary lemmas.

Lemma 1. If \( R \) is a prime ring of characteristic different from 2 and \( U \) is a Lie ideal of \( R \) such that for all \( u \in U \), \([u, d(u)] \in Z \), and \( u^2 \in U \), then
\[
[u, d(u)] = 0 \quad \text{for all } u \in U.
\]

Proof. First observe that linearizing the relation \([u, d(u)] \in Z \) on \( u = u + u^2 \), we obtain \([u^2, d(u)] + [u, ud(u) + d(u)u] \in Z \). That is, \( 4[u, d(u)]u \in Z \) for all \( u \in U \). Hence, \([u, d(u)][u, r] = 0 \) for all \( u \in U \), \( r \in R \). If for some \( u \in U \), \( [d(u), u] \neq 0 \), then, as it is in the centre \( Z \), we get \([u, r] = 0 \) for all \( r \in R \), in particular \([u, d(u)] = 0 \). Hence \([u, d(u)] = 0 \) for all \( u \in U \). \( \square \)

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Lemma 2. Let \( R \) be a prime ring and \( U \) a Lie ideal of \( R \). Suppose that \([u, d(u)] \in Z\) for all \( u \in U \). Then \([[[d(r), u], u]] = 0\) for all \( u \in U \), \( r \in R \). Further, if for all \( u \in U \), \([u, d(u)] = 0\) then \([[d(r), u]], u\] = 0 for all \( r \in R \), \( u \in U \).

Proof. Let \( u \in U \) and \( r \in R \), then \([u, r] \in U \), so that \([u+[u, r], d(u+[u, r])]\) \( \in Z \). That is, \([[u, r], d(u)]+
[u, [d(u), r]]+\[u, [u, d(r)]\] \( \in Z \). Now, \([[u, r], d(u)]+[u, [d(u), r]]\] = \([r, [d(u), u]]\) for any \( r \in R \), \( u \in U \). Since \([d(u), u] \in Z\), we get \([[u, r], d(u)]+[u, [d(u), r]] = 0\). Hence

\([[d(r), u], u] \in Z \) for all \( r \in R \), \( u \in U \).

The last part can be obtained similarly. \( \square \)

The following lemma may have some independent interest.

Lemma 3. Let \( R \) be a prime of characteristic not 2 and let \( U \) be a Jordan ideal of \( R \) with \( ud(u)\) = \( d(u)u = 0\) for all \( u \in U \). Then \( U = 0 \).

Proof. Linearize the relation \( ud(u) = 0 \) on \( u \) to get

\[ ud(v) + vd(u) = 0 \] for all \( u, v \in U \).

For \( u \in U \) and any \( r \in R \), \( u(\text{ur} - \text{ru}) + (\text{ur} - \text{ru})u \in U \). But \( 2(\text{ru}^2 - \text{u}^2r) = \{u(\text{ru} - \text{ur}) + (\text{ru} - \text{ur})u\} - \{(\text{ur} - \text{ru})u + u(\text{ur} - \text{ru})\} \). As the first and second term on the right hand side are in \( U \), \( 2(\text{ru}^2 - \text{u}^2r) \in U \). As \( 2u^2 \in U \), \( 2(\text{u}^2r + \text{ru}^2) \in U \). It follows that \( 4u^2r \) and \( 4ru^2 \) are in \( U \). Replacing \( v \) by \( 4ru^2 \) where \( r \in R \) in (1) and using the hypothesis, we get \( ud(r)u^2 = 0 \) for all \( u \in U, r \in R \). If in (1) we replace \( v \) by \( ur + ru \) where \( r \in R \), then \( u^2d(r) + ud(r)u + 2urd(u) = 0 \); and hence \( u^2d(r)u + ud(r)u^2 = 0 \). Therefore, \( u^2d(r)u = 0 \) for all \( u \in U \) and \( r \in R \). Again, put \( v = 4uru = 2\{u(\text{ru} + \text{ur}) + (\text{ru} + \text{ur})u\} - \{2u^2r + r \cdot 2u^2\} \) in (1) where \( r \in R \); then \( 0 = ud(\text{ru})ru + u^2d(r)u + u^2ru = u^2d(r)u + u^2ru = 0 \). Hence, \( u^2d(r)u + ud(r)u^2 = 0 \). Lastly, replace \( v \) by \( 4u^2r \) in (1), for \( r \in R \); then \( 0 = ud(4u^2r) + 4u^2rd(u) = 4u^3d(r) \). Hence, \( u^3d(r) = 0 \) for all \( u \in U \) and \( r \in R \). Then by Lemma 1 of [2], \( u^3 = 0 \) for all \( u \in U \). For \( u \in U \) and \( r \in R \), \( 2(\text{u}^2r + \text{ru}^2) \in U \), so that \( 0 = 2^3(\text{u}^2r + \text{ru}^2) \). Multiply on the right by \( u^2r \), to obtain \( 2^3(\text{u}^2r) = 0 \). Hence, \( (u^2r)^4 = 0 \). If for some \( u \) in \( U \), \( u^2 \neq 0 \), then \( u^2R \) is a nonzero right ideal of \( R \) in which the quartic of every element is zero. By Levitzki’s theorem [1, Lemma 1.1] \( R \) would have a nilpotent ideal; which is impossible for a prime ring. Hence \( u^2 = 0 \) for all \( u \in U \). By repeating the above argument we can show that \( u = 0 \) for all \( u \in U \). \( \square \)

3. The main theorems.

Theorem 1. Let \( R \) be a prime ring of characteristic different from 2 and 3. Let \( d \) be a nonzero derivation of \( R \), and \( U \) a Lie ideal of \( R \) with \([u, d(u)] \in Z\) for all \( u \in U \). Then \( U \subset Z \).
PROOF. By Lemma 2, \([d(r), u], u\] \(\in\) \(Z\) for all \(u \in U\), \(r \in R\). Now, proceeding on the same lines as in Posner [2] (cf. equations (16) to (27)), we have \([d(u), u] = 0\) for all \(u \in U\). Again by Lemma 2,
\[
([d(r), u], u] = 0 \quad \text{for all } u \in U, \ r \in R.
\]
Replace \(u\) by \(u + w\) with \(w \in U\) in (2).
\[
([d(r), u], w] + ([d(r), w], u] = 0 \quad \text{for all } r \in R, \ u, \ w \in U.
\]
Suppose now that \(w, \ v \in U\) are such that \(wv\) is also in \(U\). By replacing \(w\) by \(wv\) in (3), where \(v \in U\), and expanding we get
\[
w([d(r), u], v] + ([d(r), u], w)v + [d(r), w][v, u]
+ ([d(r), w], u] + w([d(r), v], u] + [w, u][d(r), v] = 0.
\]
In view of (3) the last equation reduces to \([d(r), w][v, u] + [w, u][d(r), v] = 0\). For any \(t \in R\), \(w \in U\), the element \(v = tw - wt\) satisfies the criterion \(wv \in U\), hence by above
\[
[d(r), w][t, w], u] + [w, u][d(r), [t, w]] = 0 \quad \text{for } t, r \in R; \ u, w \in U.
\]
Putting \(u = w\) in (4), we have
\[
[d(r), w][[t, w], w] = 0 \quad \text{for } r, t \in R \text{ and } w \in W.
\]
Substitution of \(td(a)\) for \(r\) in (5) with \(a \in R\) yields on expansion
\[
[d(r), w][2[t, w][d(a), w] + [[t, w], w]d(a) + t[[d(a), w], w]] = 0.
\]
By (5) the second term is zero and by (2) the third term is zero, so that
\[
[d(r), w][t, w][d(a), w] = 0 \quad \text{for all } r, t, a \in R, \ w \in U.
\]
Put \(u = [t, w]\) in (4). Then \([[t, w], w][[t, w], d(r)] = 0\). Its linearization on \(t = t + d(a)\) where \(a \in R\) together with (2) yields
\[
[[t, w], w][[d(a), w], d(r)] = 0 \quad \text{for all } a, t, r \in R \text{ and } w \in U.
\]
Replace \(t\) by \(d(t)p\) with \(p \in R\) in (7) and expand; then
\[
2[d(t), w][p, w] + d(t)[[p, w], w] + [[d(t), w], w][p][[d(a), w], d(r)] = 0.
\]
By (7) the second term is zero, while by (2) the third term is zero. Hence
\[
[d(t), w][p, w][[d(a), w], d(r)] = 0.
\]
In view of (6), the last equation reduces to
\[
[d(t), w][p, w][d(a), w] = 0 \quad \text{for all } a, r, p, t \in R \text{ and } w \in U.
\]
In (6) replace \(t\) by \(td(p)\), where \(p \in R\) and using the last equation to get
\[
[d(r), w][d(p), w][d(a), w] = 0 \quad \text{for all } r, p, a \in R \text{ and } w \in U.
\]
Now, if \([d(r), w] = 0\) for all \(r \in R\), \(w \in U\), that is for all \(r \in R\), \(w \in U\), \((I, d)r = 0\), then by \([2, \text{Theorem 1}]\) \(w \in Z\) for all \(w \in U\). Thus assume that there exists a \(w \in U\) such that for some \(r \in R\), \([d(r), w] \neq 0\). That is \(w \notin Z\). Then for all \(a, p \in R\)

\[
[d(p), w][d(a), w] = 0.^2
\]

Replace \(a\) by \(bc\) where \(c, b \in R\) and expand to get

\[
[d(p), w][d(b), w]c + [d(p), w]d(b)[c, w] + [d(p), w]b[d(c), w] + [d(p), w][b, w]d(c) = 0.
\]

Replace \(b\) by \([t, w]\) where \(t \in R\). By \((8)\) the first term is zero, while by \((6)\) the third term is zero, and by \((5)\) the fourth term is zero. Therefore,

\[
[d(p), w][d([t, w]), w] = 0.
\]

Since, \([d([t, w]), w] = [d(t), w] + [t, d(w)]\) and using \((8)\) we get

\[
[d(p), w][d([t, d(w)], w)][w, c] = 0 \quad \text{for all } p, c, t \in R \text{ and } w \in U.
\]

Replace \(c\) by \(cr_1\) where \(r_1 \in R\), then \([d(p), w][t, d(w)]R[w, c] = 0\). Since \(R\) is prime and \(w \notin Z\), we get \([d(p), w][t, d(w)] = 0\) for \(p, t \in R\), \(w \in U\). Therefore, \([d(p), w]R[t, d(w)] = 0\) for \(p, t \in R\) and \(w \in U\); which together with \([d(r), w] \neq 0\) implies that \(d(w) \in Z\).

Now suppose that \(u \in U\) and \(u \in Z\). Then \(0 = d[u, a] = [d(u), a] + [u, d(a)]\) and hence \(d(u) \in Z\). Therefore, \(d(u) \in Z\) for all \(u \in U\), so that \(d([w, a]) \in Z\) for all \(a \in R\). That is, \([d(w), a] + [w, d(a)] \in Z\) for all \(a \in R\). Thus \([w, d(a)] \in Z\) for all \(a \in R\). In particular,

\[
[w, d(aw)] = [w, d(a)]w + [w, a]d(w) \in Z.
\]

Commute \((9)\) with \(w\) to get \([w, [w, a]]d(w) = 0\) for \(a \in R\). If \(d(w) \neq 0\), and as it is in the centre \(Z\), \([w, [w, a]] = 0\) for all \(a \in R\). By \([1, \text{Sublemma}, p. 5]\) \(w \in Z\), a contradiction. Hence, \(d(w) = 0\). Thus, by \((9)\), \([w, d(a)]w \in Z\) for all \(a \in R\); that is \([d(a), w][w, b] = 0\) for \(a, b \in R\). Replace \(b\) by \(bc\), where \(c \in R\), then \([d(a), w]R[w, b] = 0\). Since \(R\) is prime, either \(w \in Z\) or \([d(a), w] = 0\) for all \(a \in R\). So, in both cases \(w \in Z\); a contradiction. Hence the conclusion is that \(w \in Z\) for all \(w \in U\). This proves the theorem. □

Now we should like to settle the problem when \(R\) has characteristic 3. Note that the assumption that the characteristic is different from 3 does not enter the proof of Theorem 1 onwards of equation \([u, d(u)] = 0\) for all \(u \in U\). Therefore, if \([u, d(u)] = 0\) holds for all \(u \in U\), we can show that \(U \subseteq Z\) even if \(R\) has characteristic 3. In view of Lemma 1, if \(R\) has characteristic different from 2 and \(U\) is a Lie ideal of \(R\) such that for all \(u \in U\),

\[
^2 \text{Onward proof of this theorem is given by the referee.}
\]
$u^2 \in U$ and $[u, d(u)] \in Z$, then $[u, d(u)] = 0$ for all $u \in U$. Hence, we get the following weaker result.

**Theorem 2.** Let $R$ be a prime ring of characteristic $3$ and $d$ a nonzero derivation of $R$. If $U$ is a Lie ideal of $R$ with $[u, d(u)] \in Z$ and $u^2 \in U$ for all $u \in U$, then $U \subseteq Z$.

Now we will show that the conclusion of Theorems 1 and 2 holds even if $U$ is a Jordan ideal of $R$. In this regard, we prove the following.

**Theorem 3.** Let $R$ be a prime ring of characteristic not equal to $2$. Let $d$ be a nonzero derivation of $R$ and $U$ be a Jordan ideal of $R$, such that $[u, d(u)] \in Z$ for all $u \in U$. Then $U \subseteq Z$.

**Proof.** For $u \in U$, $2u^2 \in U$. Therefore by Lemma 1, $[u, d(u)] = 0$ for all $u \in U$. Replace $u$ by $u + v$, where $v \in U$, then

$$[u, d(v)] + [v, d(u)] = 0 \quad \text{for all } u, v \in U.$$

In (10), replace $v$ by $ur + ru$, $r \in R$, and expand to get

$$u[u, d(r)] + [u, d(r)]u + d(u)[u, r]$$

$$+ [u, r]d(u) + u[r, d(u)] + [r, d(u)]u = 0,$$

i.e.,

$$2urd(u) - 2d(u)ru + u^2d(r) - d(r)u^2 = 0 \quad \text{for } r \in R, u \in U.$$

Replace $r$ by $ur$ in (11), then

$$d(u)(u^2r - ru^2) = 0 \quad \text{for all } r \in R, u \in U.$$

that is, $d(u)I_u(r) = 0$ for all $r \in R, u \in U$; hence by [2, Lemma 1], either

$$u^2 \in Z \quad \text{or} \quad d(u) = 0 \quad \text{for all } u \in U.$$

For $u \in U$ and any $r \in R$, $ur + ru \in U$. But

$$4uru = 2(u(ur + ru) + (ur + ru)u) - (2u^2 \cdot r + r \cdot 2u^2).$$

The first and the second term on the right are in $U$. Hence $4uru \in U$. Therefore, if we replace $v$ by $4uru$ in (10), where $r \in R$, then

$$d(u)[u, r]u + u[u, d(r)]u + u[u, r]d(u) + u[r, d(u)]u = 0,$$

i.e.,

$$u^2rd(u) - d(u)ru^2 + u^2d(r)u - ud(r)u^2 = 0 \quad \text{for } r \in R, u \in U.$$

Replace $r$ by $ur$ in (14) and use (14) to get $ud(u)(uru - ru^2) = 0$. However in view of (12), this equation reduces to $ud(u)u(ur - ru) = 0$. That is, $ud(u)u \cdot I_u(r) = 0$. By [2, Lemma 1], either

$$ud(u)u = 0 \quad \text{or} \quad u \in Z \quad \text{for all } u \in U.$$
In (12), replace $u$ by $u+v$ where $v \in U$ and use (12). Then
\[ \{d(u) + d(v); [uv + vu, r] + d(u)[v^2, r] + d(v)[u^2, r] = 0. \]
Replace $u$ by $-u$, then
\[ \{-d(u) + d(v); [-uv - vu, r] - d(u)[v^2, r] + d(v)[u^2, r] = 0. \]
Adding last two equations and dividing by 2, we have $d(u)[uv + vu, r] + d(v)[u^2, r] = 0$ for all $r \in R$ and $u, v \in U$. By Lemma 3, $ud(u) \neq 0$, for some $u$ in $U$, $d(u) \neq 0$, hence by (13) $u^2 \in Z$. The net result of this is $d(u)[uv + vu, r] = 0$.

That is, $d(u)[u^2 + vu, r] = 0$ for all $r \in R$ and $v \in U$. By [2, Lemma 1] $uv + vu \in Z$ for all $v \in U$. If $u^2 = 0$, then $0 = d(u^2) = ud(u) + d(u)u = 2ud(u)$ so that $ud(u) = 0$, a contradiction. Hence $u^2 \neq 0$. Suppose that $ud(u)u = 0$ then $u^2 d(u) = 0$ which implies that $d(u) = 0$, a contradiction. Hence $ud(u)u \neq 0$, so (15) gives $u \in Z$. Hence $2u \in Z$; so that $w \in Z$ for all $v \in U$. As $u \neq 0 \in Z$, we have $v \in Z$ for all $v \in U$. Hence $U \subseteq Z$. This completes the proof of Theorem 3.

We should like to settle the problem even when $R$ has characteristic 2. In this case Lie ideals and Jordan ideals will coincide. We are proving now the following weaker result.

**Theorem 4.** Let $R$ be a prime ring of characteristic 2, and let $d$ be a nonzero derivation of $R$. Let $U$ be a Lie (Jordan) ideal and a subring of $R$. Suppose that $[u, d(u)] \in Z$ for all $u \in U$. Then $U$ is commutative.

**Proof.** By Lemma 2, $[[d(r), u], u] \in Z$ i.e.,
\[ d(r)u^2 + u^2 d(r) \in Z \quad \text{for all } r \in R, u \in U. \]
Commuting (16) with $d(r)$ and $u^2$ respectively, we get
\[ u^2 d(r)^2 = d(r)^2 u^2 \quad \text{for all } r \in R, u \in U \]
and
\[ u^4 d(r) = d(r)u^4 \quad \text{for all } r \in R, u \in U \]
where $d(r)^2$ stands for $(d(r))^2$.

In (17a) replace $r$ by $v + u^2v$ where $v \in U$ and use (17a). Then
\[ (u^2d(v))^2 + u^2 d(v)d(u^2)v + u^2 d(u^2)vd(v) + u^4 d(v)^2 \]
\[ = (d(v)u^2)^2 + d(v)d(u^2)vu^2 + d(u^2)vd(v)u^2 + u^2 d(v)^2u^2. \]
For $u \in U$, $d(u^2) = ud(u) + d(u)u \in Z$, so that in view of (17b) the last equation reduces to $(u^2d(v) + d(v)u^2)^2 = 0$ for $u, v \in U$. Since $R$ is prime,
by using (16) we get

\[(18) \quad u^2d(v) = d(v)u^2 \quad \text{for all } u, v \in U.\]

Replace \( u \) by \( u+w \) where \( w \in U \) in (18). Then

\[(uw+wu)d(v) = d(v)(uw+wu).\]

Replace \( w \) by \( uw+wu \), then \((uw+wu)ud(v) = d(v)(uw+wu)u = (uw+wu)d(v)u.\) Therefore, \((uw+wu)(ud(v)+d(v)u) = 0 \) for all \( u, v, w \in U \). Linearize the last equation on \( u = u + u_1^2 \), where \( u_1 \in U \) and put \( v = u \). Then using (18) we get

\[ (u_1^2w + wu^2)(ud(u) + d(u)u) = 0 \quad \text{for all } u, v, w \in U. \]

If \([d(u), u] \neq 0\) for some \( u \) in \( U \), then \( u_1^2w = wu_1^2 \) for all \( u_1, w \in U \); so that, \( u^2(wr+rw) = (wr+rw)u^2 \) for all \( r \in R, u, w \in U \). That is, \( w(u^2r+ru^2) = (u^2r+ru^2)w \) for all \( r \in R, u, w \in U \). Replace \( r \) by \( ru \), then \((u^2r+ru^2)\times(wu+uw) = 0 \) for all \( r \in R, u, w \in U \). Replacing \( w \) by \([u, t] \) we get

\[ (u^2r + ru^2)(u^2t + tu^2) = 0 \quad \text{for all } r, t \in R, u, v \in U. \]

Replace \( t \) by \( tp \) where \( p \in R \); then \((u^2r+ru^2)R(u^2t+tu^2) = 0 \). Since \( R \) is prime, we get \( u^2 \in Z \) for all \( u \in U \). Thus assume that \([d(u), u] = 0 \) for all \( u \in U \). Therefore \( u^2v^2 = v^2u^2 \) for all \( u, v \in U \). Therefore \( u^2(vw + wd) = (vw + wd)u^2 \) for all \( u, v, w \in U \). Replace \( v \) by \( vw \), then \((vw + wd) \times (u^2w + wu^2) = 0 \); so that \( (w^2r+rw^2)(u^2w+wu^2) = 0 \) for all \( r \in R, u, w \in U \). The Lemma 1 of [2] forces that if \( w^2 \notin Z \) for some \( w \) in \( U \), then for that \( w, u^2w = wu^2 \) for all \( u \in U \). So that, \([u, v], w] = 0 \) for all \( u, v \in U \). For \( w \in U \), then \([v, w], u] + [[w, u], v] = [[u, v], w] = 0 \) for all \( u, v \in U \). Replace, in \([v, w], u] + [[w, u], v] = 0 \), by \( vw \) and expand to obtain \([v, w], u]w + [v, w][w, u] + [[w, u], v]w = 0 \). Hence, \([v, w][w, u] = 0 \) for all \( u, v \in U \). Replacing \( v \) by \( [w, r] \) and \( u \) by \([w, t] \), we get

\[ (w^2r + rw^2)(w^2t + tw^2) = 0 \quad \text{for all } r, t \in R. \]

Replace \( t \) by \( tp \) where \( p \in R \), then \((w^2r+rw^2)R(w^2t+tw^2) = 0 \), which implies that \( w^2 \in Z \), a contradiction. Hence the conclusion is that \( u^2 \in Z \) for all \( u \in U \). So in all possible cases \( w^2 \in Z \) for all \( u \in U \) so that \((wu+vw) \in Z \) and \((wu+uw)u \in Z \) for all \( u, v \in U \). If \( u \notin Z(U) \), where \( Z(U) \) denotes the centre of \( U \), then \( uv + vu = 0 \) for all \( v \in U \) and \( u \in Z(U) \). Hence \( U \) is commutative.
In Theorem 4, if we just assume that \( U \) is only a Lie (Jordan) ideal or only a subring of \( R \), then \( U \) may not be commutative. This is shown by the following examples.

**Example 1.** Let \( R \) be a prime ring of all \( 2 \times 2 \) matrices over a non-commutative prime ring. Consider \( U = \{(a, b) \in R\} \). It is clear that \( U \) is a subring, but not a Lie ideal of \( R \). Let us define \( d: R \to R \) such that

\[
d \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 0 & -\beta \\ \gamma & 0 \end{pmatrix}, \text{ for all } \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in R.
\]

It is easy to verify that \( d \) is a nonzero derivation of \( R \) with \([u, d(u)] \in Z\) for all \( u \in U \). But \( U \) is not commutative.

**Example 2.** Consider the prime ring \( R \) of all \( 2 \times 2 \) matrices over \( GF(2) \). Let \( U = \{(c, d) \in R\} \). It is clear that \( U \) is a Lie ideal but not a subring of \( R \). Let us define \( d: R \to R \) such that

\[
d \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d - c & a - d \\ a - d & b - c \end{pmatrix} \text{ for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R.
\]

It can be seen that \( d \) is a nonzero derivation of \( R \) with \([u, d(u)] \in Z\) for all \( u \in U \). However, \( U \) is not commutative.

Following example shows that a ring may satisfy all the assumptions of Theorem 4, but \( U \) may not be in the centre, even though \( U \) is commutative.

**Example 3.** Let \( R \) be a ring of all \( 2 \times 2 \) matrices with entries from \( GF(2) \). Consider \( U = \{(b, a) \in R\} \). It can easily be verified that \( U \) is both a Lie (Jordan) ideal and a subring of \( R \), but it is not an ideal of \( R \). Define \( d: R \to R \) as in Example 2. Then \( d \) satisfies \([u, d(u)] \in Z\) for all \( u \in U \). Clearly \( U \) is commutative, but \( U \) is not in the centre of \( R \).

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**References**