TANGENTIAL ASYMPTOTIC VALUES OF BOUNDED ANALYTIC FUNCTIONS

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Abstract. Suppose $f$ is a bounded analytic function on the unit disc whose Fatou boundary function is approximately continuous from above at 1 with value 0. It is well known that $f$ tends to zero radially and therefore along every nontangential arc. Tanaka [3] and Boehme and Weiss [1] have shown that $f$ must also tend to zero along certain arcs which are tangential from above. The purpose of this paper is to improve their results by producing a larger collection of such tangential arcs along which $f$ tends to zero. We construct a class of examples to show that our result is actually better.

1. Introduction and main theorem. Let $f$ be a bounded analytic function on the unit disc $D = \{re^{i\theta} : r < 1\}$. By a classical theorem of Fatou, $f$ has radial limits for almost all $\theta$. We denote these limits by $f(e^{i\theta})$ and call this function the (Fatou) boundary function of $f$. Given $\varepsilon > 0$ let

$$S_\varepsilon = S_\varepsilon(f) = \{e^{i\theta} : 0 \leq \theta \leq \pi, |f(e^{i\theta})| < \varepsilon\}$$

and let $S_\varepsilon'$ denote the complement of this set in the upper semicircle. Then, by definition, $f(e^{i\theta})$ is approximately continuous from above at 1 with value 0 if

$$\lim_{\theta \to 0^+} \frac{1}{\theta} \int_0^{\theta} \chi_{S_\varepsilon'}(e^{it}) \, dt = 0$$

for every $\varepsilon > 0$, where $\chi_M$ denotes the characteristic function of the set $M$. If we let $\delta(\theta) = 0^{-1} \int_0^{\theta} |f(e^{it})| \, dt$, then it is easily seen that $f$ is approximately continuous from above at 1 with value 0 if and only if $\lim_{\theta \to 0^+} \delta(\theta) = 0$.

Supposing that $\lim_{\theta \to 0^+} \delta(\theta) = 0$, Tanaka [3] proved that $f(re^{i\theta})$ tends to zero along any curve for which $2\theta(\delta(2\theta))^{1/2}/(1-r)$ remains bounded.

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This was improved in [1, Theorem 3.1], where it is shown that \( f(re^{i\theta}) \) still tends to zero along any curve for which \( 2\theta \delta(2\theta)/(1-r) \) tends to zero. Using Jensen's inequality and Lemma 3.2 of [1] we obtain the following theorem.

**Theorem 1.** Let \( f \) be a bounded analytic function on the unit disc with boundary function \( f(e^{i\theta}) \). Suppose \( f(e^{i\theta}) \) is approximately continuous from above at 1 with value 0. Then, \( f \) tends to zero along any upper tangential curve \( \Gamma \) for which

\[
\limsup_{r \to 0} \lim_{\theta \to 0^+} \frac{1}{1-r} \int_0^{2\theta} \chi_{S_{\varepsilon}}(e^{it}) \, dt = o(\log 1/\varepsilon)
\]

as \( \varepsilon \to 0 \).

**Proof.** Lemma 3.2 of [1] proves that for any measurable subset \( S \) of \( \{e^{it}: 0 \leq t \leq \pi\} \) and all small enough \( \theta > 0 \),

\[
u_S(re^{i\theta}) \geq \frac{2}{\pi} \tan^{-1}\left\{ \frac{a/2\theta}{(1-r)/\tan(\theta/2) + 4\theta[1-(a/2\theta)]/(1-r)} \right\}
\]

where \( \nu_S \) is the harmonic measure of \( S \) and

\[
a = a(S, \theta) = \int_0^{2\theta} \chi_S(e^{it}) \, dt.
\]

Thus, if \( \Gamma \) is any upper tangential arc in \( D \) terminating at 1 and if \( a(S, \theta)/2\theta \) tends to 1 as \( \theta \to 0^+ \), then

\[
\liminf_{r \to 0} \nu_S(re^{i\theta}) \geq \frac{2}{\pi} \tan^{-1}\left\{ 1/2 \limsup_{r \to 0^+, \theta \to 0} \frac{a(S', \theta)}{1-r} \right\}.
\]

We assume now without loss of generality that \( |f(e^{i\theta})| < 1 \). Using Jensen's inequality we then obtain the estimates for \( re^{i\theta} \in \Gamma, \theta \to 0, \) and \( 0 < \varepsilon < 1 \),

\[
\limsup \log |f(re^{i\theta})| \leq \limsup \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{it})| P_\varepsilon(\theta - t) \, dt
\]

\[
\leq \limsup \frac{1}{2\pi} \int_{S_{\varepsilon}} \log |f(e^{it})| P_\varepsilon(\theta - t) \, dt
\]

\[
\leq \log \varepsilon \liminf \nu_{S_{\varepsilon}}(re^{i\theta})
\]

\[
\leq \log \varepsilon \frac{2}{\pi} \tan^{-1}\left\{ 1/2 \limsup \frac{a(S', \theta)}{1-r} \right\}
\]

where \( P_\varepsilon(\theta) \) is the Poisson kernel. It is easily seen that our assumption implies that the last member of the above string of inequalities tends to \( -\infty \) along \( \Gamma \). Consequently, \( f \) must tend to zero along \( \Gamma \) as claimed.
If we now return to the condition of Theorem 3.1 of [1] we see that it implies the condition of Theorem 1. To verify this, suppose \( 2\theta \delta(2\theta)/(1-r) \) tends to zero along the upper tangential curve \( \Gamma \). Then, for any \( \epsilon > 0 \),

\[
\epsilon \cdot \frac{1}{1 - r} \int_0^{\theta} \chi_{S^2}(e^{it}) \, dt \leq \frac{1}{1 - r} \int_0^{2\theta} |f(e^{it})| \, dt = \frac{2\theta \delta(2\theta)}{1 - r}.
\]

Therefore, for any \( \epsilon > 0 \), \( \limsup_{\epsilon \to 0} a(S^2, \theta)/(1-r) = 0 \) along \( \Gamma \) and the condition of Theorem 1 above is easily satisfied.

2. A class of examples. If \( f(e^{i\theta}) \) is actually continuous from above at 1 with value 0, then for every \( \epsilon > 0 \), \( S_\epsilon \) includes an interval abutting 1 from above. Thus, the condition of Theorem 1 is satisfied for any upper tangential curve at 1 as one would hope. This is not the case for the result of [1]. It does not prove, for example, that such a continuous function must tend to zero along the upper tangential curve \( 2\theta \delta(2\theta) = 1-r \).

In this section we will construct a class of functions which demonstrates further the distance between Theorem 3.1 of [1] and Theorem 1 of this paper. Before proceeding to the construction of these examples we require some preliminaries.

Throughout the remainder of the paper \( \rho(\theta) \) will denote a nondecreasing convex function defined near 0+ with

\[
\rho(0) = \rho'(0) = 0, \quad \rho(\theta) > 0 \quad \text{for} \quad \theta \neq 0.
\]

**Lemma 1.** With \( \rho \) as above and \( c \) any constant

\[
\lim_{\theta \to 0^+} \frac{\rho(\theta + c\rho(\theta))}{\rho(\theta)} = 1.
\]

The proof of this lemma is not hard and can be found in [2].

**Lemma 2.** Let \( \rho \) be as above. Let \( c \) be a positive constant and suppose \( \theta_1 > \theta_2 > \cdots, \theta_n \to 0^+ \) are defined so that the system of intervals \( [\theta_n - c\rho(\theta_n), \theta_n + c\rho(\theta_n)] \), \( n = 1, 2, \cdots \), is nonoverlapping. Suppose further that, as \( n \to \infty \), \( \sum_{m=n+1}^{\infty} \rho(\theta_m) = o(\rho(\theta_n)) \). Then, if \( I \) is the union of the above system of intervals,

\[
\limsup_{\theta \to 0^+} \frac{1}{\rho(\theta)} \int_0^{\theta} \chi_I(e^{it}) \, dt = 2c.
\]

**Proof.** Let \( F(\theta) = (1/\rho(\theta)) \int_0^\theta \chi_I(e^{it}) \, dt. \) Clearly, when \( \theta_{n+1} + c\rho(\theta_{n+1}) \leq \theta \leq \theta_n - c\rho(\theta_n) \), we have \( F(\theta) \leq F(\theta_{n+1} + c\rho(\theta_{n+1})) \). Thus, \( \limsup F(\theta) \)
can be computed restricting \( \theta \) to lie in the intervals of \( I \). If \( \theta_n - c \rho(\theta_n) \leq \theta \leq \theta_n + c \rho(\theta_n) \), then

\[
F(\theta) = \left[ \theta - (\theta_n - c \rho(\theta_n)) + \sum_{m=n+1}^{\infty} 2c \rho(\theta_m) \right] / \rho(\theta)
\]

\[
= \frac{\rho(\theta_n + c \rho(\theta_n))}{\rho(\theta_n) - c \rho(\theta_n)} F(\theta)\]

From Lemma 1 and our hypotheses we have

\[
\lim_{\theta \to 0^+} \sup F(\theta) = \lim_{n \to \infty} \sup F(\theta_n + c \rho(\theta_n))
\]

\[
= \lim_{n \to \infty} \sup \frac{2c \rho(\theta_n) + \sum_{m=n+1}^{\infty} 2c \rho(\theta_m)}{\rho(\theta_n) - c \rho(\theta_n)}
\]

\[
= 2c.
\]

**Example.** Let \( \rho \) be a convex function as above, let \( \Gamma_\rho \) be the convex upper tangential curve \( 1-r = \rho(\theta) \). Then, there exists a corresponding bounded analytic function \( f \) such that, as \( \varepsilon \to 0^+ \),

\[
(1) \quad \lim_{\varepsilon \to 0^+} \sup_{r \in \Gamma_\rho} \frac{1}{1-r} \int_0^{2\theta} \chi_{S_\varepsilon}(e^{it}) \, dt = o(\log 1/\varepsilon)
\]

while

\[
(2) \quad \lim_{\varepsilon \to 0^+} \lim_{r \to 1^-} \sup_{e^{i\theta} \in \Gamma_\rho} \frac{1}{1-r} \int_0^{2\theta} \chi_{S_\varepsilon}(e^{it}) \, dt = \infty
\]

so that \( \lim \sup 2\theta \delta(\theta)/(1-r) \neq 0 \) as \( \theta \to 0^+ \).

**Proof.** The example is constructed in several stages. For fixed \( \varepsilon \in (0, 1/e) \) let \( m(\varepsilon) \) denote the greatest integer in \( (\log 1/\varepsilon)^{1/2} \). Choose a nondecreasing sequence \( \{a(k)\} \) of positive numbers tending to \( \infty \) but so slowly that

\[
(3) \quad \sum_{k=1}^{m(\varepsilon)} a(k) \leq (\log 1/\varepsilon)^{3/4} \quad \text{for all } 0 < \varepsilon < 1/e.
\]

For example, choose \( \{a(k)\} \) so that the average of the first \( N \) elements is less than \( \sqrt{N}/2 \).

We construct below a doubly infinite sequence of nonoverlapping intervals \( I_{n,k} = [\theta_{n,k} - a(k) \rho(\theta_{n,k}), \theta_{n,k} + a(k) \rho(\theta_{n,k})] \), \( n, k = 1, 2, 3, \ldots \) in such a way that, for each \( k \),

\[
(4) \quad \sum_{m=n+1}^{\infty} \rho(\theta_{m,k}) = o(\rho(\theta_{n,k})) \quad \text{as } n \to \infty.
\]
We, furthermore, require that if $J_0, J_1, \cdots$ are the intervals complementary to the system $\{I_{n,k}\}$ in $(0, \pi/2)$, then the right-hand endpoints $q_0, q_1, \cdots$ of $J_0, J_1, \cdots$ proceed to 0 at least as fast as a geometric sequence so that

$$\sum_{k=1}^{\infty} k^n q_k^n < \infty.$$  

Explicitly, choose $\theta_1 = \frac{1}{2}$ and positive values of $\theta_n$ such that $\theta_{n+1} < \theta_n/4^n$ and $\rho(\theta_{n+1}) < \rho(\theta_n)/4^n$. Let $\theta_n = 0^{2*(2n-1)}$, $n, k = 1, 2, 3, \cdots$; and let

$$I'_{n,k} = \{\theta : \theta'_{n,k} - a(k)\rho(\theta'_{n,k}) \leq 0 \leq \theta'_{n,k} + a(k)\rho(\theta'_{n,k})\}.$$  

For fixed $k$ and $m$ and $n=1, 2, 3, \cdots$,

$$\frac{\theta_n - a(k)\rho(\theta_n)}{\theta_{n+1} + a(m)\rho(\theta_{n+1})} \geq \frac{\theta_n - a(k)\rho(\theta_n)}{\theta_n/4^n + a(m)\rho(\theta_n/4^n)}$$

$$= \frac{4^n}{1 + a(m)\rho(\theta_n/4^n)} \rightarrow \infty \quad \text{as} \quad n \rightarrow \infty,$$

since $\rho'(0) = 0$. Because of this we may choose $n_2$ so large that the intervals $I'_{n_1,1}, I'_{n_1+1,1}, \cdots$ are nonoverlapping. Call these intervals $I_{1,1}, I_{2,1}, \cdots$ and denote their centers by $\theta_{1,1}, \theta_{2,1}, \cdots$. This done, we may choose $n_2$ so large that the intervals $I_{1,1}, I_{2,1}, \cdots$ and $I'_{n_2,2}, I'_{n_2+1,2}, \cdots$ are nonoverlapping and so that $I'_{n,2}$ is closer to zero than $I_{n,1}$. Rename the latter (primed) intervals $I_{1,2}, I_{2,2}, \cdots$ and their centers $\theta_{1,2}, \theta_{2,2}, \cdots$. Continuing inductively we obtain a system $I_{n,k}$ of nonoverlapping subintervals of $(0, \pi/2)$. It is also clear that the endpoints $q_0, q_1, \cdots$ of the complementary intervals to this system proceed to zero at least as rapidly as the midpoints $\theta_n$ of the original system. Since these latter are geometric, condition (5) above is satisfied. To verify condition (4), we first estimate

$$\sum_{m=n+1}^{\infty} \rho(\theta_m) < \sum_{j=1}^{\infty} \frac{1}{4^n + j(j - 1)/2} < \frac{1}{4^n} \frac{1}{1 - 1/4^n} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.$$  

Then, using the identity $\theta_{m,k} = \theta'_{(m+n(k-1))}, k = 0^{2*(2m+2n(k-3))}$, which follows from our construction, we have for fixed $k$

$$\sum_{m=n+1}^{\infty} \rho(\theta_{m,k}) \leq \sum_{m=2k(2n+2n-1)}^{\infty} \rho(\theta_{m,k}) \rho(\theta_{2n+2(n-3)}) \rightarrow 0$$

where $N = 2k(2n+2n-1)$.

The next step is to define the function $f$ of the example. For $k=1, 2, \cdots$ let $J_k = \bigcup_{n=1}^{\infty} J_{n,k}$. Define $k(\theta)$ for $0 \leq \theta \leq 2\pi$ by setting $k(\theta) = 0$ for $\theta \in [\pi/2, 2\pi]$ and for $0 \in J_0$. For $k=1, 2, \cdots$ let $k(\theta) = -k^2$ for $0 \in I_k \cup J_k$.  

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Since by condition (5) above $-\int_0^{2\pi} k(\theta) \, d\theta \leq \sum_{k=1}^{\infty} k^2 \varphi_k < \infty$, we have that $k(\theta)$ is integrable. Therefore, the function
\[
f(z) = \exp \frac{1}{2\pi} \int_0^{2\pi} e^{is} \frac{e^{it} + z}{e^{it} - z} k(t) \, dt
\]
is bounded analytic on $D$ and as is well known, $|f(e^{i\theta})|=e^{k(\theta)}$ a.e.

We now see that for $0<\varepsilon<1/e$, $|f(e^{i\theta})|\geq \varepsilon$ if and only if $e^{k(\theta)}\geq \varepsilon$ or $k(\theta)\geq \ln \varepsilon$. Since the values of $k(\theta)$ are $0, -1, -4, -9, \ldots$ we have $-k^2\leq \ln \varepsilon$ if and only if $k\leq m(\varepsilon)$. Thus, for each $0<\varepsilon<1/e$ there is an interval $K_{\varepsilon}$ abutting 0 on the right such that $S_{\varepsilon} \cap K_{\varepsilon}$ consists exactly of all but a finite number of the intervals of $\bigcup_{k=1}^{m(\varepsilon)} I_k$.

We are now in a position to verify the properties (1) and (2) of the example. Recalling Lemma 2 and the definitions of $a(k)$, $m(\varepsilon)$, $S_{\varepsilon}^c$, $T_{\rho}$, and $I_k$ we make the following estimates for $\theta \to 0^+$, $\theta \in K_{\varepsilon}$ and $re^{i\theta} \in T_{\rho}$,
\[
2a(m(\varepsilon)) = \lim \sup \frac{1}{\rho(2\theta)} \int_0^{2\theta} \chi_{I_{m(\varepsilon)}}(e^{it}) \, dt
= \lim \sup \frac{1}{1-r} \int_0^{2\theta} \chi_{I_{m(\varepsilon)}}(e^{it}) \, dt
\leq \lim \sup \frac{1}{1-r} \int_0^{2\theta} \chi_{S_{\varepsilon}^c}(e^{it}) \, dt
= \lim \sup \sum_{k=1}^{m(\varepsilon)} \frac{1}{\rho(2\theta)} \int_0^{2\theta} \chi_{I_k}(e^{it}) \, dt \leq 2 \sum_{k=1}^{m(\varepsilon)} a(k)
\leq (\log 1/\varepsilon)^{3/4} = o(\log 1/\varepsilon).
\]

Finally, the assertion following (2) of the example is seen from the computations of the last paragraph of $\S1$.

REFERENCES


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