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TANGENTIAL ASYMPTOTIC VALUES OF BOUNDED ANALTYIC FUNCTIONS

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ABSTRACT. Suppose f is a bounded analytic function on the unit disc whose Fatou boundary function is approximately continuous from above at 1 with value 0. It is well known that f tends to zero radially and therefore along every nontangential arc. Tanaka [3] and Boehme and Weiss [1] have shown that f must also tend to zero along certain arcs which are tangential from above. The purpose of this paper is to improve their results by producing a larger collection of such tangential arcs along which f tends to zero. We construct a class of examples to show that our result is actually better.

1. **Introduction and main theorem.** Let f be a bounded analytic function on the unit disc $D=\{re^{i\theta}:r<1\}$. By a classical theorem of Fatou, f has radial limits for almost all θ. We denote these limits by $f(e^{i\theta})$ and call this function the (Fatou) boundary function of f. Given $\varepsilon>0$ let

$$S_\varepsilon = S_\varepsilon(f) = \{e^{i\theta}:0\leqq\theta\leqq\pi,\ |f(e^{i\theta})| < \varepsilon\}$$

and let S'_ε denote the complement of this set in the upper semicircle. Then, by definition, $f(e^{i\theta})$ is approximately continuous from above at 1 with value 0 if

$$\lim_{\theta\to 0^+} \frac{1}{\theta}\int_0^\theta \chi_{S'_\varepsilon}(e^{it})\,dt = 0$$

for every $\varepsilon>0$, where χ_M denotes the characteristic function of the set M. If we let $\delta(\theta)=\theta^{-1}\int_0^\theta |f(e^{it})|\,dt$, then it is easily seen that f is approximately continuous from above at 1 with value 0 if and only if $\lim_{\theta\to 0^+}\delta(\theta)=0$.

Supposing that $\lim_{\theta\to 0^+}\delta(\theta)=0$, Tanaka [3] proved that $f(re^{i\theta})$ tends to zero along any curve for which $2\theta(\delta(2\theta))^{1/2}/(1-r)$ remains bounded.

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This was improved in [**1**, Theorem 3.1], where it is shown that $f(re^{i\theta})$ still tends to zero along any curve for which $2\theta\delta(2\theta)/(1-r)$ tends to zero. Using Jensen's inequality and Lemma 3.2 of [**1**] we obtain the following theorem.

THEOREM 1. *Let f be a bounded analytic function on the unit disc with boundary function $f(e^{i\theta})$. Suppose $f(e^{i\theta})$ is approximately continuous from above at* 1 *with value* 0. *Then, f tends to zero along any upper tangential curve Γ for which*

$$\limsup_{re^{i\theta}\to 1, re^{i\theta}\in\Gamma} \frac{1}{1-r}\int_0^{2\theta} \chi_{S'_\varepsilon}(e^{it})\,dt = o(\log 1/\varepsilon)$$

as $\varepsilon\to 0$.

PROOF. Lemma 3.2 of [**1**] proves that for any measurable subset S of $\{e^{it}: 0 \leqq t \leqq \pi\}$ and all small enough $\theta > 0$,

$$u_S(re^{i\theta}) \geqq \frac{2}{\pi}\tan^{-1}\left\{\frac{a/2\theta}{(1-r)/\tan(\theta/2) + 4\theta[1-(a/2\theta)]/(1-r)}\right\}$$

where u_S is the harmonic measure of S and

$$a = a(S,\theta) = \int_0^{2\theta} \chi_S(e^{it})\,dt.$$

Thus, if Γ is any upper tangential arc in D terminating at 1 and if $a(S,\theta)/2\theta$ tends to 1 as $\theta\to 0^+$, then

$$\liminf_{re^{i\theta}\to 1, re^{i\theta}\in\Gamma} u_S(re^{i\theta}) \geqq \frac{2}{\pi}\tan^{-1}\left\{1\Big/ 2\limsup_{re^{i\theta}\to 1, re^{i\theta}\in\Gamma}\frac{a(S',\theta)}{1-r}\right\}.$$

We assume now without loss of generality that $|f(e^{i\theta})| < 1$. Using Jensen's inequality we then obtain the estimates for $re^{i\theta}\in\Gamma$, $\theta\to 0$, and $0<\varepsilon<1$,

$$\begin{aligned}\limsup \log|f(re^{i\theta})| &\leqq \limsup \frac{1}{2\pi}\int_0^{2\pi} \log|f(e^{it})|\,P_r(\theta-t)\,dt \\ &\leqq \limsup \frac{1}{2\pi}\int_{S_\varepsilon} \log|f(e^{it})|\,P_r(\theta-t)\,dt \\ &\leqq \log\varepsilon \liminf u_{S_\varepsilon}(re^{i\theta}) \\ &\leqq \log\varepsilon\,\frac{2}{\pi}\tan^{-1}\left\{1\Big/ 2\limsup\frac{a(S'_\varepsilon,\theta)}{1-r}\right\}\end{aligned}$$

where $P_r(\theta)$ is the Poisson kernel. It is easily seen that our assumption implies that the last member of the above string of inequalities tends to $-\infty$ along Γ. Consequently, f must tend to zero along Γ as claimed.

If we now return to the condition of Theorem 3.1 of [1] we see that it implies the condition of Theorem 1. To verify this, suppose $2\theta\delta(2\theta)/(1-r)$ tends to zero along the upper tangential curve Γ. Then, for any $\varepsilon>0$,

$$\varepsilon\cdot\frac{1}{1-r}\int_0^{2\theta}\chi_{S'_\varepsilon}(e^{it})\,dt \leqq \frac{1}{1-r}\int_0^{2\theta}|f(e^{it})|\,dt = \frac{2\theta\delta(2\theta)}{1-r}.$$

Therefore, for any $\varepsilon>0$, $\limsup a(S'_\varepsilon,\theta)/(1-r)=0$ along Γ and the condition of Theorem 1 above is easily satisfied.

2. **A class of examples.** If $f(e^{i\theta})$ is actually continuous from above at 1 with value 0, then for every $\varepsilon>0$, S_ε includes an interval abutting 1 from above. Thus, the condition of Theorem 1 is satisfied for any upper tangential curve at 1 as one would hope. This is not the case for the result of [1]. It does not prove, for example, that such a continuous function must tend to zero along the upper tangential curve $2\theta\delta(2\theta)=1-r$.

In this section we will construct a class of functions which demonstrates further the distance between Theorem 3.1 of [1] and Theorem 1 of this paper. Before proceeding to the construction of these examples we require some preliminaries.

Throughout the remainder of the paper $\rho(\theta)$ will denote a nondecreasing convex function defined near 0^+ with

$$\rho(0)=\rho'(0)=0, \qquad \rho(\theta)>0 \quad \text{for } \theta\neq 0.$$

LEMMA 1. *With ρ as above and c any constant*

$$\lim_{\theta\to 0^+}\frac{\rho(\theta+c\rho(\theta))}{\rho(\theta)}=1.$$

The proof of this lemma is not hard and can be found in [2].

LEMMA 2. *Let ρ be as above. Let c be a positive constant and suppose $\theta_1>\theta_2>\cdots$, $\theta_n\to 0^+$ are defined so that the system of intervals $[\theta_n-c\rho(\theta_n),\theta_n+c\rho(\theta_n)]$, $n=1,2,\cdots$, is nonoverlapping. Suppose further that, as $n\to\infty$, $\sum_{m=n+1}^{\infty}\rho(\theta_m)=o(\rho(\theta_n))$. Then, if I is the union of the above system of intervals,*

$$\limsup_{\theta\to 0^+}\frac{1}{\rho(\theta)}\int_0^{\theta}\chi_I(e^{it})\,dt = 2c.$$

PROOF. Let $F(\theta)=(1/\rho(\theta))\int_0^\theta\chi_I(e^{it})\,dt$. Clearly, when $\theta_{n+1}+c\rho(\theta_{n+1})\leqq\theta\leqq\theta_n-c\rho(\theta_n)$, we have $F(\theta)\leqq F(\theta_{n+1}+c\rho(\theta_{n+1}))$. Thus, $\limsup F(\theta)$

can be computed restricting θ to lie in the intervals of I. If $\theta_n - c\rho(\theta_n) \leqq \theta \leqq \theta_n + c\rho(\theta_n)$, then

$$F(\theta) = \left[\theta - (\theta_n - c\rho(\theta_n)) + \sum_{m=n+1}^{\infty} 2c\rho(\theta_m)\right] \Big/ \rho(\theta)$$

$$\leqq \left[2c\rho(\theta_n) + \sum_{m=n+1}^{\infty} 2c\rho(\theta_m)\right] \Big/ \rho(\theta_n - c\rho(\theta_n))$$

$$= \frac{\rho(\theta_n + c\rho(\theta_n))}{\rho(\theta_n)} \frac{\rho(\theta_n)}{\rho(\theta_n - c\rho(\theta_n))} F(\theta_n + c\rho(\theta_n)).$$

From Lemma 1 and our hypotheses we have

$$\limsup_{\theta \to 0^+} F(\theta) = \limsup_{n \to \infty} F(\theta_n + c\rho(\theta_n))$$

$$= \limsup_{n \to \infty} \frac{2c\rho(\theta_n) + \sum_{m=n+1}^{\infty} 2c\rho(\theta_m)}{\rho(\theta_n + c\rho(\theta_n))}$$

$$= 2c.$$

EXAMPLE. Let ρ be a convex function as above, let Γ_ρ be the convex upper tangential curve $1 - r = \rho(2\theta)$. Then, there exists a corresponding bounded analytic function f such that, as $\varepsilon \to 0^+$,

$$\limsup_{re^{i\theta} \to 1, re^{i\theta} \in \Gamma_\rho} \frac{1}{1-r} \int_0^{2\theta} \chi_{S'_\varepsilon}(e^{it})\, dt = o(\log 1/\varepsilon) \tag{1}$$

while

$$\lim_{\varepsilon \to 0^+} \limsup_{re^{i\theta} \to 1, re^{i\theta} \in \Gamma_\rho} \frac{1}{1-r} \int^{2\theta} \chi_{S'_\varepsilon}(e^{it})\, dt = \infty \tag{2}$$

so that lim sup $2\theta\delta(2\theta)/(1-r) \neq 0$ as $\theta \to 0^+$.

PROOF. The example is constructed in several stages. For fixed $\varepsilon \in (0, 1/e)$ let $m(\varepsilon)$ denote the greatest integer in $(\log 1/\varepsilon)^{1/2}$. Choose a nondecreasing sequence $\{a(k)\}$ of positive numbers tending to ∞ but so slowly that

$$2\sum_{k=1}^{m(\varepsilon)} a(k) \leqq (\log 1/\varepsilon)^{3/4} \quad \text{for all } 0 < \varepsilon < 1/e. \tag{3}$$

For example, choose $\{a(k)\}$ so that the average of the first N elements is less than $\sqrt{N}/2$.

We construct below a doubly infinite sequence of nonoverlapping intervals $I_{n,k} = [\theta_{n,k} - a(k)\rho(\theta_{n,k}), \theta_{n,k} + a(k)\rho(\theta_{n,k})]$, $n, k = 1, 2, 3, \cdots$ in such a way that, for each k,

$$\sum_{m=n+1}^{\infty} \rho(\theta_{m,k}) = o(\rho(\theta_{n,k})), \quad \text{as } n \to \infty. \tag{4}$$

We, furthermore, require that if $J_0, J_1, \cdots$ are the intervals complementary to the system $\{I_{n,k}\}$ in $(0, \pi/2)$, then the right-hand endpoints $\varphi_0, \varphi_1, \cdots$ of $J_0, J_1, \cdots$ proceed to 0 at least as fast as a geometric sequence so that

$$\sum_{k=1}^{\infty} k^2 \varphi_k < \infty. \tag{5}$$

Explicitly, choose $\theta_1 = \frac{1}{2}$ and positive values of θ_n such that $\theta_{n+1} < \theta_n/4^n$ and $\rho(\theta_{n+1}) < \rho(\theta_n)/4^n$. Let $\theta'_{n,k} = \theta_{2^k(2n-1)}$, $n, k = 1, 2, 3, \cdots$; and let

$$I'_{n,k} = \{\theta : \theta'_{n,k} - a(k)\rho(\theta'_{n,k}) \leqq \theta \leqq \theta'_{n,k} + a(k)\rho(\theta'_{n,k})\}.$$

For fixed k and m and $n = 1, 2, 3, \cdots$,

$$\begin{aligned} \frac{\theta_n - a(k)\rho(\theta_n)}{\theta_{n+1} + a(m)\rho(\theta_{n+1})} &> \frac{\theta_n - a(k)\rho(\theta_n)}{\theta_n/4^n + a(m)\rho(\theta_n/4^n)} \\ &= 4^n \frac{1 - a(k)\rho(\theta_n)/\theta_n}{1 + a(m)\rho(\theta_n/4^n)/(\theta_n/4^n)} \to \infty \quad \text{as } n \to \infty, \end{aligned}$$

since $\rho'(0) = 0$. Because of this we may choose n_1 so large that the intervals $I'_{n_1,1}, I'_{n_1+1,1}, \cdots$ are nonoverlapping. Call these intervals $I_{1,1}, I_{2,1}, \cdots$ and denote their centers by $\theta_{1,1}, \theta_{2,1}, \cdots$. This done, we may choose n_2 so large that the intervals $I_{1,1}, I_{2,1}, \cdots$ and $I'_{n_2,2}, I'_{n_2+1,2}, \cdots$ are nonoverlapping and so that $I'_{n_2,2}$ is closer to zero than $I_{1,1}$. Rename the latter (primed) intervals $I_{1,2}, I_{2,2}, \cdots$ and their centers $\theta_{1,2}, \theta_{2,2}, \cdots$. Continuing inductively we obtain a system $I_{n,k}$ of nonoverlapping subintervals of $(0, \pi/2)$. It is also clear that the endpoints $\varphi_0, \varphi_1, \cdots$ of the complementary intervals to this system proceed to zero at least as rapidly as the midpoints θ_n of the original system. Since these latter are geometric, condition (5) above is satisfied. To verify condition (4), we first estimate

$$\frac{\sum_{m=n+1}^{\infty} \rho(\theta_m)}{\rho(\theta_n)} < \sum_{j=1}^{\infty} \frac{1}{4^{jn} + j(j-1)/2} < \frac{1}{4^n} \frac{1}{1 - 1/4^n} \to 0 \quad \text{as } n \to \infty.$$

Then, using the identity $\theta_{m,k} = \theta'_{(m+n_k-1),k} = \theta_{2^k(2m+2n_k-3)}$, which follows from our construction, we have for fixed k

$$\frac{\sum_{m=n+1}^{\infty} \rho(\theta_{m,k})}{\rho(\theta_{n,k})} \leqq \frac{\sum_{m=2^k(2n+2n_k-1)}^{\infty} \rho(\theta_m)}{\rho(\theta_{2^k(2n+2n_k-3)})} < \frac{\sum_{m=N}^{\infty} \rho(\theta_m)}{\rho(\theta_{N-1})} \to 0$$

where $N = 2^k(2n + 2n_k - 1)$.

The next step is to define the function f of the example. For $k = 1, 2, \cdots$ let $I_k = \bigcup_{n=1}^{\infty} I_{n,k}$. Define $k(\theta)$ for $0 \leqq \theta \leqq 2\pi$ by setting $k(\theta) = 0$ for $\theta \in [\pi/2, 2\pi]$ and for $\theta \in J_0$. For $k = 1, 2, \cdots$ let $k(\theta) = -k^2$ for $\theta \in I_k \cup J_k$.

Since by condition (5) above $-\int_0^{2\pi} k(\theta)\, d\theta \leqq \sum_{k=1}^{\infty} k^2\varphi_k < \infty$, we have that $k(\theta)$ is integrable. Therefore, the function

$$f(z) = \exp \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\, k(t)\, dt$$

is bounded analytic on D and as is well known, $|f(e^{i\theta})| = e^{k(\theta)}$ a.e.

We now see that for $0<\varepsilon<1/e$, $|f(e^{i\theta})| \geqq \varepsilon$ if and only if $e^{k(\theta)} \geqq \varepsilon$ or $k(\theta) \geqq \ln \varepsilon$. Since the values of $k(\theta)$ are $0, -1, -4, -9, \cdots$ we have $-k^2 \geqq \ln \varepsilon$ if and only if $k \leqq m(\varepsilon)$. Thus, for each $0<\varepsilon<1/e$ there is an interval K_ε abutting 0 on the right such that $S'_\varepsilon \cap K_\varepsilon$ consists exactly of all but a finite number of the intervals of $\bigcup_{k=1}^{m(\varepsilon)} I_k$.

We are now in a position to verify the properties (1) and (2) of the example. Recalling Lemma 2 and the definitions of $a(k)$, $m(\varepsilon)$, S'_ε, Γ_ρ, and I_k we make the following estimates for $\theta \to 0^+$, $\theta \in K_\varepsilon$ and $re^{i\theta} \in \Gamma_\rho$,

$$\begin{aligned}
2a(m(\varepsilon)) &= \limsup \frac{1}{\rho(2\theta)} \int_0^{2\theta} \chi_{I_{m(\varepsilon)}}(e^{it})\, dt \\
&= \limsup \frac{1}{1-r} \int_0^{2\theta} \chi_{I_{m(\varepsilon)}}(e^{it})\, dt \\
&\leqq \limsup \frac{1}{1-r} \int_0^{2\theta} \chi_{S'_\varepsilon}(e^{it})\, dt \\
&= \limsup \sum_{k=1}^{m(\varepsilon)} \frac{1}{\rho(2\theta)} \int_0^{2\theta} \chi_{I_k}(e^{it})\, dt \leqq 2 \sum_{k=1}^{m(\varepsilon)} a(k) \\
&\leqq (\log 1/\varepsilon)^{3/4} = o(\log 1/\varepsilon).
\end{aligned}$$

Finally, the assertion following (2) of the example is seen from the computations of the last paragraph of §1.

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