TANGENTIAL ASYMPTOTIC VALUES OF BOUNDED ANALYTIC FUNCTIONS

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Abstract. Suppose \( f \) is a bounded analytic function on the unit disc whose Fatou boundary function is approximately continuous from above at 1 with value 0. It is well known that \( f \) tends to zero radially and therefore along every nontangential arc. Tanaka [3] and Boehme and Weiss [1] have shown that \( f \) must also tend to zero along certain arcs which are tangential from above. The purpose of this paper is to improve their results by producing a larger collection of such tangential arcs along which \( f \) tends to zero. We construct a class of examples to show that our result is actually better.

1. Introduction and main theorem. Let \( f \) be a bounded analytic function on the unit disc \( D = \{ re^{i\theta} : r < 1 \} \). By a classical theorem of Fatou, \( f \) has radial limits for almost all \( \theta \). We denote these limits by \( f(e^{i\theta}) \) and call this function the (Fatou) boundary function of \( f \). Given \( \varepsilon > 0 \) let

\[ S_\varepsilon = S_\varepsilon(f) = \{ e^{i\theta} : 0 \leq \theta \leq \pi, |f(e^{i\theta})| < \varepsilon \} \]

and let \( S'_\varepsilon \) denote the complement of this set in the upper semicircle. Then, by definition, \( f(e^{i\theta}) \) is approximately continuous from above at 1 with value 0 if

\[ \lim_{\varepsilon \to 0^+} \frac{1}{\theta} \int_0^\theta \chi_{S'_\varepsilon}(e^{it}) \, dt = 0 \]

for every \( \varepsilon > 0 \), where \( \chi_M \) denotes the characteristic function of the set \( M \). If we let \( \delta(\theta) = -1 \int_0^\theta \frac{1}{\theta} |f(e^{it})| \, dt \), then it is easily seen that \( f \) is approximately continuous from above at 1 with value 0 if and only if \( \lim_{\theta \to 0^+} \delta(\theta) = 0 \).

Supposing that \( \lim_{\theta \to 0^+} \delta(\theta) = 0 \), Tanaka [3] proved that \( f(re^{i\theta}) \) tends to zero along any curve for which \( 2\theta(\delta(2\theta))^{1/2}/(1-r) \) remains bounded.
This was improved in [1, Theorem 3.1], where it is shown that $f(re^{i\theta})$ still tends to zero along any curve for which $2\theta\delta(2\theta)/(1-r)$ tends to zero. Using Jensen’s inequality and Lemma 3.2 of [1] we obtain the following theorem.

**Theorem 1.** Let $f$ be a bounded analytic function on the unit disc with boundary function $f(e^{i\theta})$. Suppose $f(e^{i\theta})$ is approximately continuous from above at 1 with value 0. Then, $f$ tends to zero along any upper tangential curve $\Gamma$ for which

$$
\limsup_{re^{i\theta} \to 1, re^{i\theta} \in \Gamma} \frac{1}{1-r} \int_0^{2\theta} \chi_{S_r}(e^{it}) \, dt = o(\log 1/\varepsilon)
$$

as $\varepsilon \to 0$.

**Proof.** Lemma 3.2 of [1] proves that for any measurable subset $S$ of $\{e^{it}: 0 \leq t \leq \pi\}$ and all small enough $\theta > 0$,

$$u_S(re^{i\theta}) \geq \frac{2}{\pi} \tan^{-1}\left(\frac{a/2\theta}{(1-r)/\tan(\theta/2) + 4\theta[1-(a/2\theta)]/(1-r)}\right),$$

where $u_S$ is the harmonic measure of $S$ and

$$a = a(S, \theta) = \int_0^{2\theta} \chi_S(e^{it}) \, dt.$$

Thus, if $\Gamma$ is any upper tangential arc in $D$ terminating at 1 and if $a(S, \theta)/2\theta$ tends to 1 as $\theta \to 0^+$, then

$$\liminf_{re^{i\theta} \to 1, re^{i\theta} \in \Gamma} u_S(re^{i\theta}) \geq \frac{2}{\pi} \tan^{-1}\left(\frac{1/2 \limsup_{re^{i\theta} \to 1, re^{i\theta} \in \Gamma} a(S', \theta)}{1 - r}\right).$$

We assume now without loss of generality that $|f(e^{i\theta})| < 1$. Using Jensen’s inequality we then obtain the estimates for $re^{i\theta} \in \Gamma$, $\theta \to 0$, and $0 < \varepsilon < 1$,

$$\limsup \log |f(re^{i\theta})| \leq \limsup_{\theta \to 0} \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{it})| P_r(\theta - t) \, dt \leq \limsup_{\theta \to 0} \frac{1}{2\pi} \int_{S_r} \log |f(e^{it})| P_r(\theta - t) \, dt \leq \log \varepsilon \liminf u_{S_r}(re^{i\theta}) \leq \log \varepsilon \frac{2}{\pi} \tan^{-1}\left(\frac{1/2 \limsup a(S', \theta)}{1 - r}\right)$$

where $P_r(\theta)$ is the Poisson kernel. It is easily seen that our assumption implies that the last member of the above string of inequalities tends to $-\infty$ along $\Gamma$. Consequently, $f$ must tend to zero along $\Gamma$ as claimed.
If we now return to the condition of Theorem 3.1 of [1] we see that it implies the condition of Theorem 1. To verify this, suppose \( 2\theta \delta(2\theta)/(1-r) \) tends to zero along the upper tangential curve \( \Gamma \). Then, for any \( \varepsilon > 0 \),

\[
\varepsilon \cdot \frac{1}{1-r} \int_0^{2\theta} \chi_{S_{\varepsilon t}}(e^{it}) \, dt \leq \frac{1}{1-r} \int_0^{2\theta} |f(e^{it})| \, dt = \frac{2\theta \delta(2\theta)}{1-r}.
\]

Therefore, for any \( \varepsilon > 0 \), \( \lim \sup a(S_{\varepsilon t}, \theta)/(1-r)=0 \) along \( \Gamma \) and the condition of Theorem 1 above is easily satisfied.

2. A class of examples. If \( f(e^{i\theta}) \) is actually continuous from above at 1 with value 0, then for every \( \varepsilon > 0 \), \( S_{\varepsilon} \) includes an interval abutting 1 from above. Thus, the condition of Theorem 1 is satisfied for any upper tangential curve at 1 as one would hope. This is not the case for the result of [1]. It does not prove, for example, that such a continuous function must tend to zero along the upper tangential curve \( 2\theta \delta(2\theta)=1-r \).

In this section we will construct a class of functions which demonstrates further the distance between Theorem 3.1 of [1] and Theorem 1 of this paper. Before proceeding to the construction of these examples we require some preliminaries.

Throughout the remainder of the paper \( \rho(\theta) \) will denote a nondecreasing convex function defined near \( 0^+ \) with

\[
\rho(0) = \rho'(0) = 0, \quad \rho(\theta) > 0 \quad \text{for} \quad \theta \neq 0.
\]

**Lemma 1.** With \( \rho \) as above and \( c \) any constant

\[
\lim_{\theta \to 0^+} \frac{\rho(\theta + c\rho(\theta))}{\rho(\theta)} = 1.
\]

The proof of this lemma is not hard and can be found in [2].

**Lemma 2.** Let \( \rho \) be as above. Let \( c \) be a positive constant and suppose \( \theta_1 > \theta_2 > \cdots, \theta_n \to 0^+ \) are defined so that the system of intervals \([\theta_n-c\rho(\theta_n), \theta_n+c\rho(\theta_n)]\), \( n=1, 2, \cdots \), is nonoverlapping. Suppose further that, as \( n \to \infty \), \( \sum_{n=m}^{\infty} \rho(\theta_n)=o(\rho(\theta_n)) \). Then, if \( I \) is the union of the above system of intervals,

\[
\lim_{\theta \to 0^+} \frac{1}{\rho(\theta)} \int_0^\theta \chi_I(e^{it}) \, dt = 2c.
\]

**Proof.** Let \( F(\theta)=(1/\rho(\theta)) \int_0^\theta \chi_I(e^{it}) \, dt \). Clearly, when \( \theta_{n+1}+c\rho(\theta_{n+1}) \leq \theta \leq \theta_n-c\rho(\theta_n), \) we have \( F(\theta) \leq F(\theta_{n+1}+c\rho(\theta_{n+1})) \). Thus, \( \lim \sup F(\theta) \)
can be computed restricting \( \theta \) to lie in the intervals of \( I \). If \( \theta_n - c \rho(\theta_n) \leq \theta \leq \theta_n + c \rho(\theta_n) \), then

\[
F(\theta) = \left[ \theta - (\theta_n - c \rho(\theta_n)) + \sum_{m=n+1}^{\infty} 2c \rho(\theta_m) \right] / \rho(\theta)
\]

\[
\leq \left[ 2c \rho(\theta_n) + \sum_{m=n+1}^{\infty} 2c \rho(\theta_m) \right] / \rho(\theta_n - c \rho(\theta_n))
\]

\[
= \frac{\rho(\theta_n + c \rho(\theta_n))}{\rho(\theta_n)} \frac{\rho(\theta_n)}{\rho(\theta_n - c \rho(\theta_n))} F(\theta_n + c \rho(\theta_n)).
\]

From Lemma 1 and our hypotheses we have

\[
\limsup_{\theta \to 0^+} F(\theta) = \limsup_{n \to \infty} F(\theta_n + c \rho(\theta_n)) = \limsup_{n \to \infty} \frac{2c \rho(\theta_n) + \sum_{m=n+1}^{\infty} 2c \rho(\theta_m)}{\rho(\theta_n + c \rho(\theta_n))} = 2c.
\]

Example. Let \( \rho \) be a convex function as above, let \( \Gamma_\rho \) be the convex upper tangential curve \( 1 - r = \rho(2\theta) \). Then, there exists a corresponding bounded analytic function \( f \) such that, as \( \varepsilon \to 0^+ \),

\[
(1) \quad \limsup_{\varepsilon \to 0^+} \int_{r \to -1, r \to 1} \chi_{S(\varepsilon)}(e^{it}) \, dt = o(\log 1/\varepsilon)
\]

while

\[
(2) \quad \limsup_{e \to 0^+} \limsup_{\varepsilon \to 0^+} \int_{r \to -1, r \to 1} \chi_{S(\varepsilon)}(e^{it}) \, dt = \infty
\]

so that \( \limsup 2\theta \delta(2\theta)/(1 - r) \neq 0 \) as \( \theta \to 0^+ \).

Proof. The example is constructed in several stages. For fixed \( \varepsilon \in (0, 1/\varepsilon) \) let \( m(\varepsilon) \) denote the greatest integer in \( (\log 1/\varepsilon)^{1/2} \). Choose a nondecreasing sequence \( \{a(k)\} \) of positive numbers tending to \( \infty \) but so slowly that

\[
2 \sum_{k=1}^{m(\varepsilon)} a(k) \leq (\log 1/\varepsilon)^{3/4} \quad \text{for all } 0 < \varepsilon < 1/\varepsilon.
\]

For example, choose \( \{a(k)\} \) so that the average of the first \( N \) elements is less than \( \sqrt{N}/2 \).

We construct below a doubly infinite sequence of nonoverlapping intervals \( I_{n,k} = [\theta_{n,k} - a(k) \rho(\theta_{n,k}), \theta_{n,k} + a(k) \rho(\theta_{n,k})] \), \( n, k = 1, 2, 3, \ldots \) in such a way that, for each \( k \),

\[
\sum_{m=n+1}^{\infty} \rho(\theta_{m,k}) = o(\rho(\theta_{n,k})), \quad \text{as } n \to \infty.
\]
We, furthermore, require that if $J_0, J_1, \cdots$ are the intervals complementary to the system $\{I_{n,k}\}$ in $(0, \pi/2)$, then the right-hand endpoints $\varphi_0, \varphi_1, \cdots$ of $J_0, J_1, \cdots$ proceed to 0 at least as fast as a geometric sequence so that

\[
\sum_{k=1}^{\infty} k^2 q_k^2 < \infty.
\]

Explicitly, choose $\theta_0 = \frac{1}{2}$ and positive values of $\theta_n$ such that $\theta_n < n/4^n$ and $\rho(\theta_{n+1}) < \rho(\theta_n)^{4^n}$. Let $\theta_n \in (2\pi (2n-1)), n, k = 1, 2, 3, \cdots$; and let

\[
I_{n,k} = \{ \theta : \theta_n - a(k)\rho(\theta_n) \leq \theta_n + a(k)\rho(\theta_n) \}.
\]

For fixed $k$ and $m$ and $n = 1, 2, 3, \cdots$,

\[
\frac{\theta_n - a(k)\rho(\theta_n)}{\theta_n + a(m)\rho(\theta_n)} > \frac{\theta_n - a(k)\rho(\theta_n)}{\theta_n/4^n + a(m)\rho(\theta_n/4^n)} = \frac{4^n - 1 - a(k)\rho(\theta_n)/\theta_n}{1 + a(m)\rho(\theta_n/4^n)/(\theta_n/4^n)} \to \infty \quad \text{as} \quad n \to \infty,
\]

since $\rho'(0) = 0$. Because of this we may choose $n_1$ so large that the intervals $I'_{n_1,1}, I'_{n_1+1,1}, \cdots$ are nonoverlapping. Call these intervals $I_{1,1}, I_{2,1}, \cdots$ and denote their centers by $\theta_1, \theta_2, \cdots$. This done, we may choose $n_2$ so large that the intervals $I_{1,1}, I_{2,1}, \cdots$ and $I'_{n_2,2}, I'_{n_2+1,2}, \cdots$ are nonoverlapping and so that $I'_{n_2,2}$ is closer to zero than $I_{1,1}$. Rename the latter (primed) intervals $I_{1,2}, I_{2,2}, \cdots$ and their centers $\theta_{1,2}, \theta_{2,2}, \cdots$. Continuing inductively we obtain a system $I_{n,k}$ of nonoverlapping subintervals of $(0, \pi/2)$. It is also clear that the endpoints $\varphi_0, \varphi_1, \cdots$ of the complementary intervals to this system proceed to zero at least as rapidly as the midpoints $\theta_n$ of the original system. Since these latter are geometric, condition (5) above is satisfied. To verify condition (4), we first estimate

\[
\sum_{m=n+1}^{\infty} \rho(\theta_m) \rho(\theta_n)^{-1} \to 0 \quad \text{as} \quad n \to \infty.
\]

Then, using the identity $\theta_{m,k} = \theta_{m+n-1,k} = \theta_{2k(2n+2n-3)}$, which follows from our construction, we have for fixed $k$

\[
\sum_{m=n+1}^{\infty} \rho(\theta_{m,k}) \rho(\theta_{n,k})^{-1} \leq \sum_{m=2k(2n+2n-3)}^{\infty} \rho(\theta_{m,k}) \rho(\theta_{2k(2n+2n-3)})^{-1} \to 0
\]

where $N = 2^k(2n+2n-1)$.

The next step is to define the function $f$ of the example. For $k = 1, 2, \cdots$ let $I_k = \bigcup_{n=1}^{\infty} I_{n,k}$. Define $k(\theta)$ for $0 \leq \theta \leq 2\pi$ by setting $k(\theta) = 0$ for $\theta \in [\pi/2, 2\pi]$ and for $\theta \in J_0$. For $k = 1, 2, \cdots$ let $k(\theta) = -k^2$ for $\theta \in I_k \cup J_k$.
Since by condition (5) above \(-\int_0^{2\pi} k(\theta) \, d\theta \leq \sum_{k=1}^{\infty} k^2 \varphi_k < \infty\), we have that \(k(\theta)\) is integrable. Therefore, the function

\[
f(z) = \exp \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{e^{it} + z}{e^{it} - z} \right) k(t) \, dt
\]

is bounded analytic on \(D\) and as is well known, \(|f(e^{i\theta})| = e^{k(\theta)}\) a.e.

We now see that for \(0 < \varepsilon < 1/e\), \(|f(e^{i\theta})| \geq \varepsilon\) if and only if \(e^{k(\theta)} \geq \ln \varepsilon\). Since the values of \(k(\theta)\) are \(0, -1, -4, -9, \ldots\) we have \(-k^2 \geq \ln \varepsilon\) if and only if \(k \leq m(\varepsilon)\). Thus, for each \(0 < \varepsilon < 1/e\) there is an interval \(K_\varepsilon\) abutting 0 on the right such that \(S' \cap K_\varepsilon\) consists exactly of all but a finite number of the intervals of \(\bigcup_{k=1}^{m(\varepsilon)} I_k\).

We are now in a position to verify the properties (1) and (2) of the example. Recalling Lemma 2 and the definitions of \(a(k), m(\varepsilon), S'_\varepsilon, \Gamma_\rho, \) and \(I_k\) we make the following estimates for \(\theta \to 0^+, \theta \in K_\varepsilon\) and \(re^{i\theta} \in \Gamma_\rho,\)

\[
2a(m(\varepsilon)) = \lim \sup \frac{1}{\rho(2\theta)} \int_0^{2\theta} \chi_{I_m(\varepsilon)}(e^{it}) \, dt
\]

\[
= \lim \sup \frac{1}{1 - r} \int_0^{2\theta} \chi_{I_m(\varepsilon)}(e^{it}) \, dt
\]

\[
\leq \lim \sup \frac{1}{1 - r} \int_0^{2\theta} \chi_{S'_\varepsilon}(e^{it}) \, dt
\]

\[
= \lim \sup \sum_{k=1}^{m(\varepsilon)} \frac{1}{\rho(2\theta)} \int_0^{2\theta} \chi_{I_k}(e^{it}) \, dt \leq 2 \sum_{k=1}^{m(\varepsilon)} a(k)
\]

\[
\leq (\log 1/\varepsilon)^{3/4} = o(\log 1/\varepsilon).
\]

Finally, the assertion following (2) of the example is seen from the computations of the last paragraph of §1.

REFERENCES