PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 41, Number 1, November 1973

COMPLEX ITERATED RADICALS

LEON GERBER

ABSTRACT. We prove the convergence of the sequence S defined by $z_{n+1}=(z_n-c)^{1/2}$, c real, for any choice of z_0. Let $k=|\frac{1}{4}-c|^{1/2}$. If $c<0$ or $c=\frac{1}{4}$, S has only one fixed point $w=\frac{1}{2}+k$ and converges to w for any z_0. If $0\leqq c<\frac{1}{4}$, S has the fixed points $w_1=\frac{1}{2}+k$ and $w_2=\frac{1}{2}-k$, and for any $z_0\neq w_2$, S converges to w_1. If $c>\frac{1}{4}$, S has the fixed points $w_1=\frac{1}{2}+ik$ and $w_2=\frac{1}{2}-ik$ and converges to w_1 if $\operatorname{Re}(z_0)\geqq 0$ and to w_2 otherwise. We show that convergence is strictly monotone when the neighborhood system is the pencil of coaxial circles with w_1 and w_2 as limiting points, and give rates of convergence.

The purpose of this paper is to prove that the sequence of complex numbers defined by $z_{n+1}=(z_n-c)^{1/2}$, c real, converges for any choice of z_0, i.e. globally, and to discuss the limit and rate of convergence. This problem was posed by C. S. Ogilvy [**1**].

As a guide to global convergence we first investigate local convergence. The following facts are known about the sequence defined by $z_{n+1}=f(z_n)$ for some choice of z_0.

LEMMA 1. *If $\lim_{n\to\infty} z_n=w$ and f is continuous at w then $f(w)=w$, i.e. w is a fixed point.*

PROOF. $w=\lim_{n\to\infty} z_{n+1}=\lim_{n\to\infty} f(z_n)=f(w)$.

Suppose w is a fixed point of f. If $z_N=w$ for some N, then $z_n=w$ for all $n\geqq N$ and we call the sequence *trivial*. We define

$$q(n, w)=|z_{n+1}-w|/|z_n-w|.$$

LEMMA 2. *Suppose w is a fixed point of f and $f'(w)$ exists. If $|f'(w)|<1$, then for z_0 sufficiently close to w the sequence converges to w and for all nontrivial sequences $\lim_{n\to\infty} q(n, w)=|f'(w)|$ and we say that $|f'(w)|$ is the local rate of convergence. If $|f'(w)|>1$, then the sequence cannot converge to w except trivially.*

Received by the editors June 15, 1972 and, in revised form, February 13, 1973.

AMS (MOS) subject classifications (1970). Primary 40A05, 41A25; Secondary 39A10, 50D45.

Key words and phrases. Iterated radicals, convergence, rate of convergence, coaxial circles.

PROOF. The result for trivial sequences is immediate. Otherwise $q(n, w)=|z_{n+1}-w|/|z_n-w|=|f(z_n)-f(w)|/|z_n-w|$. If $|f'(w)|=1-2\varepsilon$ for some $\varepsilon>0$, then $q(n, w)<1-\varepsilon$ for z_n in some deleted circular neighborhood $\mathscr{M}$ of w by the definition of derivative. Since $\mathscr{M}$ is circular, z_{n+1} also lies in $\mathscr{M}$. For z_0 in $\mathscr{M}$ it follows that $|z_n-w|<|z_0-w|(1-\varepsilon)^n$. The conclusion is immediate.

If $|f'(w)|=1+2\varepsilon$ for some $\varepsilon>0$ then $q(n, w)>1+\varepsilon$ for z_n in some deleted neighborhood $\mathscr{M}$ of w so that the sequence is not eventually in $\mathscr{M}$.

We now focus on the sequence defined by $z_{n+1}=(z_n-c)^{1/2}$ where the argument of the square root lies in $(-\frac{1}{2}\pi, \frac{1}{2}\pi]$. We note that $\mathrm{Re}(z_{n+1})\geqq 0$ for $n\geqq 0$ and that if w is a fixed point, $z_N=w$ implies $z_{N-1}=w$ and thus $z_0=w$ so we exclude the cases $\mathrm{Re}(z_0)<0$ and $z_0=w$ from further discussion. We define $k=|\frac{1}{4}-c|^{1/2}$.

LEMMA 3. *If $c<\frac{1}{4}$, the sequence converges locally to $w_1=\frac{1}{2}+k$ at the rate $(1+2k)^{-1}$. For $0\leqq c<\frac{1}{4}$, $w_2=\frac{1}{2}-k$ is the other fixed point but the sequence cannot converge to w_2.*

If $c>\frac{1}{4}$, the sequence converges locally to $w_1=\frac{1}{2}+ki$ and $w_2=\frac{1}{2}-ki$ at the rate $(1+4k^2)^{-1/2}$.

PROOF. The roots of $w^2=w-c$ are $w_1+(\frac{1}{4}-c)^{1/2}$ and $w_2=\frac{1}{2}-(\frac{1}{4}-c)^{1/2}$ and these are the roots of $w=(w-c)^{1/2}$ except when $c<0$ in which case $w_2<0$. Since $f(z)=(z-c)^{1/2}$, $f'(w)=\frac{1}{2}(w-c)^{1/2}=\frac{1}{2}w$.

If $c<\frac{1}{4}$, $f'(w_1)=(1+2k)^{-1}<1$.

If $0<c<\frac{1}{4}$, $f'(w_2)=(1-2k)^{-1}>1$.

If $c>\frac{1}{4}$, $|f'(w_1)|=|f'(w_2)|=\frac{1}{2}c^{1/2}=(1+4k^2)^{-1/2}<1$. □

We shall now prove global convergence and obtain rates very nearly equal to the local rates of convergence. We first consider two simple cases.

THEOREM 1. *For $c<0$, the sequence converges to w_1 for all z_0 and $q(n, w_1)\leqq(\frac{1}{2}+k)^{-1}$. For $c>1$, the sequence converges to w_1 if $\mathrm{Im}(z_0)\geqq 0$ and to w_2 if $\mathrm{Im}(z_0)<0$. In each case $q(n, w)\leqq(\frac{1}{4}+k^2)^{-1/2}$.*

PROOF. For any c and either w,

$$1/q(n, w) = |z_n - w|/|z_{n+1} - w| = |z_{n+1} + w|.$$

For $c<0$, $w_1=\frac{1}{2}+k$ is real and $\mathrm{Re}(z_{n+1})\geqq 0$ so $|z_{n+1}+w_1|\geqq w_1$. For $c>\frac{1}{4}$ and a fortiori for $c>1$, $w_1=\frac{1}{2}+ki$ lies in the first quadrant and if z_0 lies in the closed first quadrant so does z_n for each n so

$$|z_{n+1} + w_1| \geqq |w_1| = c^{1/2} = (\tfrac{1}{4} + k^2)^{1/2}.$$

The argument is similar if $\mathrm{Im}(z_0)<0$. □

The factor in the first case is twice the local rate; the factor in the second case will be improved in Theorem 3.

If $\frac{1}{4}<c<1$, $z_n=c+2\varepsilon^2 i$, and $\varepsilon>0$ is sufficiently small, then

$$|z_{n+1}+w| = |\varepsilon+\varepsilon i+w| < |w|+2\varepsilon = c^{1/2}+2\varepsilon < 1.$$

If $0<c<\frac{1}{4}$ and $c<z_n<w_2$, then $0<z_{n+1}<z_n$ and again $q(n, w)>1$. Since, as we shall prove, every sequence converges, we see that a global convergence factor cannot be expressed in terms of the usual metric in which the distance of z_n from w is the radius of the circle passing through z_n with w as *center*. Instead, for each value of c, we set up a metric based on pencils of coaxial circles with w_1 and w_2 as *limiting points*.

For $0\leqq c\leqq\frac{1}{4}$ we have $w_1=\frac{1}{2}+k$ and $w_2=\frac{1}{2}-k$ where $0\leqq k\leqq\frac{1}{2}$. The centers of these circles lie on the real axis outside the interval $(\frac{1}{2}-k, \frac{1}{2}+k)$ and the circle with center $\frac{1}{2}\pm H$ has radius R where $R^2+k^2=H^2$. The line $x=\frac{1}{2}$ is the radical axis of the pencil. For $c=\frac{1}{4}$, $w_1=w_2=\frac{1}{2}$ and the circles are tangent to $x=\frac{1}{2}$ at $(\frac{1}{2}, 0)$. For each z not on the radical axis there is precisely one circle $C(z)$ of the pencil passing through z. These are the only properties of the pencil we require. We define $d(z, w_1)$ to be the radius of $C(z)$ if $\mathrm{Re}(z)>\frac{1}{2}$ and ∞ otherwise, and $d(z, w_2)$ to be the radius of $C(z)$ if $\mathrm{Re}(z)<\frac{1}{2}$ and ∞ otherwise. Notice that $d(z, w)$ and $|z-w|$ are asymptotic for z near w if $c\neq\frac{1}{4}$.

THEOREM 2. *For $0\leqq c\leqq\frac{1}{4}$, $d(z_n, w_2)$ is a strictly increasing function of n as long as it is finite and $d_n=d(z_n, w_1)$ is eventually finite. Then d_n is strictly decreasing to zero. In fact*

$$\begin{array}{ll} d_{n+1} \leqq (\tfrac{1}{2}d_n)^{1/2} & \textit{for } d_n \geqq \tfrac{1}{2}, \\ 1/d_{n+1} \geqq 1/d_n + 1 & \textit{for } d_n \leqq 1, \\ 1/d_{n+1} \geqq 1/d_n + 2 - 4d_n & \textit{for } d_n \leqq \tfrac{1}{4}, \textit{ and} \\ d_{n+1} \leqq (1+4k)^{-1/2} d_n & \textit{for } c < \tfrac{1}{4} \textit{ and all } d_n < \infty. \end{array}$$

COMMENTS. For $c=\frac{1}{4}$ the next to the last inequality gives the correct asymptotic rate of convergence. For $0\leqq c<\frac{1}{4}$ the global convergence factor of $(1+4k)^{-1/2}$ is never more than $2\cdot 3^{-1/2}=1.15$ times the local convergence factor $(1+2k)^{-1}$. When we restrict $\mathrm{Re}(z_n)\geqq\frac{1}{2}$ the global factor for $c<0$ of Theorem 1 improves to $(1+k)^{-1}$.

PROOF. Suppose $d(z_{n+1}, w_2)=R<\infty$ i.e.

$$z_{n+1} = \tfrac{1}{2} - H + R\cos\theta + iR\sin\theta = u + iR\sin\theta$$

where $H=(R^2+k^2)^{1/2}$ and $0\leqq u<\frac{1}{2}$. Let $r=aR$ and $h=(r^2+k^2)^{1/2}$. We shall show for the proper choice of $a<1$ that $|z_n-(\frac{1}{2}-h)|<r$, i.e. $d(z_n, w_2)<r$.

This implies that $d(z_{n+1}, w_2) > a^{-1} d(z_n, w_2)$. In fact

$$\begin{aligned}|z_n - (\tfrac{1}{2} - h)|^2 - r^2 &= |z_{n+1}^2 + c - \tfrac{1}{2} + h|^2 - r^2 \\ &= |(u + iR\sin\theta)^2 - k^2 - \tfrac{1}{4} + h|^2 - r^2 \\ &= |(2uR\cos\theta + h - H) + 2uRi\sin\theta|^2 - r^2,\end{aligned}$$

after substituting for u^2 and $k^2 = H^2 - R^2$,

$$\begin{aligned}&= (2uR\cos\theta + h - H)^2 + (2uR\sin\theta)^2 - r^2 \\ &= 4(R\cos\theta - H)[R^2(u + \tfrac{1}{2}) - (H - h)(u + H)],\end{aligned} \tag{1}$$

after substituting for u^2 again. The first factor is negative Since

$$H - h = \frac{H^2 - h^2}{H + h} = \frac{R^2 - r^2}{H + h} = \frac{R^2(1 - a^2)}{H + h},$$

the second factor equals

$$\begin{aligned}R^2(u + \tfrac{1}{2}) - \frac{R^2(1 - a^2)(u + H)}{H + h}& \\ &> R^2[u + \tfrac{1}{2} - (1 - a^2)(u + \tfrac{3}{4})/2k], \quad \text{for } H \leqq \tfrac{3}{4}, \\ &> 0 \quad \text{for } a = (1 - 4k/3)^{1/2}.\end{aligned}$$

Thus for $0 \leqq c < \frac{1}{4}$ the relation $d(z_{n+1}, w_2) > (1 - 4k/3)^{-1/2} d(z_n, w_2)$ is valid until H exceeds $\frac{3}{4}$. If $c = \frac{1}{4}$ then $H = R$ and $h = r = aR$ and the second factor rearranges to $R[(a + R - 1)u + (a - \frac{1}{2})R]$ which is positive for $a = \max(1 - R, \frac{1}{2})$. Here too H eventually exceeds $\frac{3}{4}$. Now the preimage of the line $x = \frac{1}{2}$ under the map $f(z) = (z - c)^{1/2}$ is the parabola $x = c + \frac{1}{4} - y^2$ and the portion of this parabola in the right half-plane is interior to the circle corresponding to $H = \frac{3}{4}$, i.e. the circle with center $-\frac{1}{4}$ and radius $(c + 5/16)^{1/2}$. Thus once an iterate has a value of H exceeding $\frac{3}{4}$ it lies outside the parabola and the next iterate lies to the right of $x = \frac{1}{2}$. This proves the first assertion.

When $d(z_{n+1}, w_1) = R < \infty$ we need to show for the proper choice of $a > 1$ that $|z_n - (\frac{1}{2} + h)| > r$. Changing the signs of H and h in (1) and rearranging we get

$$\begin{aligned}&|z_n - (\tfrac{1}{2} + h)|^2 - r^2 \\ &\quad = 2(R\cos\theta + H)[2R^2(R\cos\theta + H + 1) - (h - H)(2R\cos\theta + 1)].\end{aligned}$$

The first factor is positive and $R\cos\theta + H + 1 > 2R\cos\theta + 1$ so all we require is $2R^2 \geqq h - H$. Now $(2R^2 + H)^2 - h^2 = R^2(4R^2 + 4H + 1 - a^2)$. Since $R \leqq H$ the expression is nonnegative for $a = 2R + 1$ which implies $r = R(2R + 1)$ so $R < (r/2)^{1/2}$. By the quadratic formula $1/R \geqq [1 + (1 + 8r)^{1/2}]/2r$ which yields the next two results. We get the final result with $a^2 = 1 + 4k < 1 + 4h$.

□

If $c>\frac{1}{4}$ then $w_1=\frac{1}{2}+ki$ and $w_2=\frac{1}{2}-ki$. The centers of the circles of the pencil lie on $x=\frac{1}{2}$ and the circles with center $\frac{1}{2}\pm Hi$ have radius R where $R^2+k^2=H^2$. The real axis is the radical axis. For each z not on the radical axis there is precisely one circle $C(z)$ of the pencil passing through z. We define $d(z, w_1)$ to be the radius of $C(z)$ if $\operatorname{Im}(z)>0$ and ∞ otherwise and $d(z, w_2)$ to be the radius of $C(z)$ if $\operatorname{Im}(z)<0$ and ∞ otherwise.

THEOREM 3. *For $c>\frac{1}{4}$, if z_0 is real then $d(z_n, w_1)$ is eventually finite. In fact, for a fixed $z_0>\frac{1}{2}$ and c near $\frac{1}{4}$, the number of iterations required is asymptotic to π/k. If $0<d(z_n, w_i)<\infty$, then*

$$d(z_{n+1}, w_i) \leqq (1+k^2)^{-1/2} d(z_n, w_i), \qquad i=1,2.$$

PROOF. If z_{n+1} is real, then $z_{n+1}=(z_n-c)^{1/2}\leqq(z_n-c)+\frac{1}{4}=z_n-(c-\frac{1}{4})$. Thus some iterate is less than c and the next lies in the first quadrant. If $z_0<2^{2^m}$ then clearly $z_n<2$ for some $n\leqq m$. To estimate the number N of steps required for z to move from 2 to c for small values of k, observe that

$$z_{n+1}-z_n=z_{n+1}-(z_{n+1}^2+c)=-(z_{n+1}-\tfrac{1}{2})^2-(c-\tfrac{1}{4}),$$

so if we let $z_n=\frac{1}{2}-kx_n$, x ranges from $-3/2k$ to $(\frac{1}{4}-k^2)/k$ and $x_{n+1}-x_n=(x_{n+1}^2+1)k$. Let p be a positive integer. If $x_n\geqq 2^j p$ then $x_{n+1}-x_n>2^{2j}p^2k$ so that the number of steps for x to move from 2^jp to $2^{j+1}p$ is less than

$$(2^{j+1}p-2^jp)/2^{2j}p^2k+1=1/2^jpk+1, \qquad j=0,1,\cdots.$$

Thus the total number of steps required to move from p to $(\frac{1}{4}-k^2)/k$ is less than $2/pk+\log_2[(\frac{1}{4}-k^2)/pk]<3/pk$. A similar argument shows that the number of steps from $-3/2k$ to $-p$ is also bounded by $3/pk$. Now take $x_0=-p$ and $x_M=p$, i.e. M is the number of steps from $-p$ to p. Then

$$\sum_{n=0}^{M-1}\frac{x_{n+1}-x_n}{x_{n+1}^2+1}=Mk.$$

The left side is a Riemann sum for $\int_{-p}^{p} dx/(x^2+1)$ and since $x_{n+1}-x_n\leqq(p^2+1)k$, the norm approaches zero with k. Clearly,

$$2\tan^{-1}p<\lim_{k\to 0}Mk<2\tan^{-1}p+\frac{6}{p}<\pi+\frac{6}{p}.$$

Since p can be arbitrarily large we have the result. Computer calculations show that if $c=0.250001$ i.e. $k=0.001$ and $z_0=2$, then $z_{3139}<c<z_{3138}$.

Suppose now $d(z_{n+1}, w_1)=R<\infty$, i.e.

$$z_{n+1}=(\tfrac{1}{2}+Hi)+R\cos\theta+iR\sin\theta=\tfrac{1}{2}+R\cos\theta+iv,$$

where $v=H+R\sin\theta>0$. We shall show that $d(z_n, w_1)\geqq(1+k^2)^{1/2}R=r$ i.e. that $|z_n-(\frac{1}{2}+hi)|\geqq r$ where $h^2=r^2+k^2$. In fact,

$$\begin{aligned}|z_n - (\tfrac{1}{2} + hi)|^2 - r^2 \\ &= |z_{n+1}^2 + c - (\tfrac{1}{2} + hi)|^2 - r^2 \\ &= |(\tfrac{1}{2} + R\cos\theta + iv)^2 + (H^2 - R^2 + \tfrac{1}{4}) - (\tfrac{1}{2} + hi)|^2 - r^2 \\ &= |R(\cos\theta - 2v\sin\theta) + [R(\sin\theta + 2v\cos\theta) + H - h]i|^2 - r^2,\end{aligned}$$

substituting for v^2,

$$\begin{aligned}&= R^2(\cos\theta - 2v\sin\theta)^2 + [R(\sin\theta + 2v\cos\theta) + H - h]^2 - r^2 \\ &= 2v[2vR^2 - (1 + 2R\cos\theta)(h - H)],\end{aligned}$$

using $h^2-r^2=H^2-R^2$. The first factor is positive. The second rearranges to

$$\begin{aligned}2HR^2 - (h - H) + 2R[R^2\sin\theta - (h - H)\cos\theta] \\ \geqq 2HR^2 - (h - H) - 2R[R^4 + (h - H)^2]^{1/2},\end{aligned}$$

via the Schwartz inequality,

$$= R^2(h + H)^{-1}\{2H(h + H) - k^2 - 2R[(h + H)^2 + k^4]^{1/2}\}$$

since $h-H=(h^2-H^2)/(h+H)=k^2R^2/(h+H)$. Further, $2H(h+H)-k^2\geqq 3k^2>0$ so positivity follows from

$$\begin{aligned}[2H(h + H) - k^2]^2 - 4R^2[(h + H)^2 + k^4] \\ = k^2(4k^2R^2 + 4Hh + 5k^2)>0.\end{aligned}$$

If $d(z_0, w_2)<\infty$, the convergence of z to w_2 is identical to the convergence of $\bar{z}$ to w_1.

Reference

1. C. S. Ogilvy, *To what limits do complex iterated radicals converge?*, Amer. Math. Monthly 77 (1970), 388–389.

Department of Mathematics, St. John's University, Jamaica, New York 11432