INDEX OF FREDHOLM OPERATORS

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Abstract. Let $X, Y, Z$ be Banach spaces, and let $T: X \rightarrow Y$ and $S: Y \rightarrow Z$ be Fredholm operators. Let $\text{ind}(T)$ denote the index of $T$. A short proof is given for the identity $\text{ind}(ST) = \text{ind}(S) + \text{ind}(T)$.

We give a “one-diagram” proof of the following theorem [3, Theorem 3, p. 121]. The notation of [3] will be used.

Theorem. Let $X, Y, Z$ be Banach spaces, and $T: X \rightarrow Y$ and $S: Y \rightarrow Z$ be Fredholm operators. Then

$$\text{ind}(ST) = \text{ind}(S) + \text{ind}(T).$$

Proof. By [3, Corollary 3, p. 121], $ST$ is a Fredholm operator. It is easy to verify directly that the outer perimeter sequence in the following commutative diagram of Banach spaces is exact [1, p. 25].

Now notice that if $0 \rightarrow A_0 \rightarrow A_1 \rightarrow \cdots \rightarrow A_n \rightarrow 0$ is an exact sequence of finite dimensional Banach spaces, then

$$\sum_{i=0}^{n} (-1)^i \dim(A_i) = 0.$$ 

The desired equation follows immediately.

Corollary 1. Let $F: X \rightarrow X$ be a Fredholm operator. Then, if for $n \geq 0$, $\text{ind}(T^n) = 0$, then $\text{ind}(T) = 0$.

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Corollary 2. Let $K: X \to X$ be a compact operator, and $I: X \to X$ be the identity operator. Then, $\text{ind}(I+K)=0$.

Proof. This follows from [2, (11.3.3), p. 321] and Corollary 1.

Bibliography


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