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# THE TANGENT BUNDLE OF AN $H$-MANIFOLD

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ABSTRACT. By an $H$-manifold we mean a closed, smooth ($C^\infty$) manifold which is an $H$-space. It is proved that the tangent sphere bundle of an $H$-manifold is fiber homotopy equivalent to the trivial bundle. This improves a result of W. Browder and E. Spanier which proved only the stable fiber homotopy triviality. As an application, we observe that a 1-connected, finite, CW complex, which is an $H$-space (and, hence, an $n$-dimensional Poincaré complex, for some $n$) is of the homotopy type of a parallelizable manifold, if $n \neq 4k+2$.

1. **Introduction.** Let $M^m$ be a closed, smooth, $m$-dimensional manifold which admits an $H$-space product, $m: M \times M \to M$, with unit $e$. We will call $M$ an $H$-manifold. In [**4**], W. Browder and E. Spanier prove that the tangent sphere bundle of $M$, $T_\circ M$, is stably fiber homotopy trivial. The purpose of this note is to prove that $T_\circ M$ is actually fiber homotopy trivial.

This is the strongest assertion one can make in general, in the following sense: there is an example of an $H$-manifold, $M$, with a nonzero rational Pontrjagin class [**1**]. This means that $TM$ is not even stably trivial as a Euclidean bundle [**14**].

2. **Main results.** In [**11**], J. Wagoner and R. Benlian prove that the fiber homotopy type of the tangent sphere bundle of a smooth manifold is a homotopy type invariant.

We observe, first, that the techniques of [**11**], also prove the following:

THEOREM 1 (WAGONER AND BENLIAN). *Let $W^k$ and $N^n$ be closed, smooth manifolds, with $2n \leqq k$, and $n \geqq 3$. Let $f$ and $g$ be embeddings of $N$ into $W$. Suppose there is a homotopy equivalence, $F: W \to W$, such that $Ff \simeq g$.*

$$\begin{tikzcd} W \arrow[rr, "F"] & & W \\ & N \arrow[ul, "f"] \arrow[ur, "g"'] & \end{tikzcd}$$

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*Assume, further, that $f$ and $g$ induce monomorphisms on the fundamental group. Then the normal sphere bundles of $f$ and $g$ are fiber homotopy equivalent.*

PROOF. (For completeness, we state the two main ideas of the proof. The details of the proof are identical with those of Proposition 1 of [**11**].) First one gets, via codimension one surgery [**12**], and an embedding theorem of Haefliger [**8**], an $h$-cobordism $V^{k+1}$, with $\partial V = W \cup W_1$, and an embedding $g_1: N \to W_1$, such that $f$ is homotopic to $g_1$ in $V$. By the methods of [**9**], it is shown that the normal sphere bundles of $g_1$ and $g$ are fiber homotopy equivalent.

The second step is to observe that $f$ and $g_1$ extend to an immersion of $N \times I$ into $V$, so that their normal sphere bundles are equivalent. This is shown by putting the homotopy between $f$ and $g_1$ into general position and using a theorem of Whitney [**13**], to redefine it in neighborhoods of the singular points, so that the new homotopy is an immersion of $N \times I$ in $V$. We refer the reader to [**11**] for a more complete version.

The main result now follows easily.

THEOREM 2. *Let $M^m$ be a connected, closed, smooth H-manifold. Then the tangent sphere bundle of $M$ is fiber homotopy trivial.*

PROOF. We may assume $m \geqq 3$, since the only closed connected $H$-manifolds of dimension 1 or 2 are $S^1$ and $S^1 \times S^1$, and they are parallelizable. Let $m: M \times M \to M$ be an $H$-space product, with unit $e$. Apply Theorem 1, with $W = M \times M$, $f: M \to M \times M$ defined by $f(x) = (x, e)$, $g: M \to M \times M$ the diagonal map, and $F: M \times M \to M \times M$ defined by $F(x, y) = (x, m(x, y))$. Note that the dimension hypotheses of Theorem 1 are satisfied, and $F$ is a homotopy equivalence [**15**]. Moreover, $Ff = g$ and, both $f$ and $g$ induce monomorphisms on the fundamental group. Therefore, the normal sphere bundles of $f$ and $g$ are fiber homotopy equivalent. But, the normal bundle of $g$ is the tangent bundle of $M$, and the normal bundle of $f$ is trivial.

3. **Remarks.** (a) According to [**4**], [**7**], and [**10**], there is a single obstruction to the fiber homotopy triviality of the tangent sphere bundle of an $H$-manifold. In even dimensions it vanishes since the Euler characteristic of a closed $H$-manifold is zero (unless it is a point) [**10**]. In odd dimensions, the obstruction vanishes if the mod 2 Kervaire semicharacteristic of $M$

$$\chi^*(M) \equiv \sum_{i=0}^{[k/2]} \dim H^i(M; Z_2) \mod 2,$$

is zero [7]. The geometric proof in Theorem 1 and Theorem 2 show this to be the case.

(b) If $M$ is 1-connected, with dimension greater than five, and an $H$-space, but only a PL or topological manifold, then the tangent microbundle of $M$ is fiber homotopy trivial.

4. **Application.** Although it is known that a finite CW complex, $X$, which is an $H$-space, is not necessarily homotopy equivalent to a compact Lie group, we can use Theorem 2 to get a reasonably nice representative of its homotopy type.

According to [**2**], there exists an $n$ such that $X$ is an $n$ dimensional Poincaré complex. From [**4**], we see that the product vector bundle over $X$ is reducible and is therefore stably fiber homotopy equivalent to the Spivak normal fibration. In this situation, simply connected surgery yields:

THEOREM 3. *Let $X$ be a 1-connected finite $H$-complex. Suppose $X$ has dimension $n$ as a Poincaré complex, and $n \neq 4k+2$. Then $X$ is of the homotopy type of a parallelizable manifold.*

PROOF. First assume $n \geqq 5$. It will follow from [**3**] that $X$ is homotopy equivalent to a $\pi$-manifold, $M$, if the surgery obstruction vanishes when $n=4k$. But this is equivalent to the index of $X$, $I(X)$, being zero, which is so because $X$ is an $H$-space [**6**]. (This can be seen easily—for $I(X)= I(S_1 \times \cdots \times S_l)$, each $S_j$ an odd dimensional sphere. But the index is multiplicative and zero on odd dimensional manifolds.) Thus, the obstruction vanishes. By Theorem 2, the tangent sphere bundle of $M$ is fiber homotopy trivial. By Sutherland [**10**], this is sufficient to make $M$ parallelizable.

For $1 \leqq n \leqq 4$, we note that $X$ will be a 1-connected finite complex with vanishing Euler characteristic. None can exist in dimensions 1 and 2. In dimension 3 or 4, $X$ would be a homology sphere. But a homology 4-sphere has the wrong cohomology for an $H$-space, and by Browder [**5**], a 1-connected homology 3-sphere is of the homotopy type of $S^3$, a parallelizable manifold.

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