ON  \( \sum_{n \leq x} \sigma^*(n) \) AND  \( \sum_{n \leq x} \varphi^*(n) \)

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Abstract. Let \( \sigma^*(n) \) and \( \varphi^*(n) \) be the unitary analogues of \( \sigma(n) \) and \( \varphi(n) \) respectively. It is known that \( E(x) = \sum_{n \leq x} \sigma^*(n) - \left( \frac{\pi^2 x^2}{12 \zeta(3)} \right) = O(x \log^2 x) \) and

\[
F(x) = \sum_{n \leq x} \varphi^*(n) - \frac{1}{2} ax^2 = O(x \log^2 x),
\]

where \( a \) is a positive constant. In this paper we improve the order estimates of \( E(x) \) and \( F(x) \) to \( E(x) = O(x \log^5 x) \) and \( F(x) = O(x \log^{5/3} x (\log \log x)^{1/3}) \).

1. Introduction. It is well known that a divisor \( d > 0 \) of the positive integer \( n \) is called unitary, if \( d \delta = n \) and \( (d, \delta) = 1 \). Let \( \sigma^*(n) \) denote the sum of the unitary divisors of \( n \). For integers \( a, b, \beta > 0 \), let \( (a, b)^* \) denote the greatest divisor of \( a \) which is a unitary divisor of \( b \). If \( (a, b)^* = 1 \), then \( a \) is said to be semiprime to \( b \). The symbol \( (a, b)^* \) was introduced by E. Cohen (cf. [1, §1]). Let \( \varphi^*(n) \) denote the number of positive integers \( a \leq n \) such that \( a \) is semiprime to \( n \).

E. Cohen (cf. [1, Corollaries 4.1.1 and 4.1.2]) established the following asymptotic formulae:

\[
(1.1) \quad \sum_{n \leq x} \sigma^*(n) = \frac{\pi^2 x^2}{12 \zeta(3)} + O(x \log^2 x),
\]

where \( \zeta(s) \) denotes the Riemann Zeta function defined by \( \zeta(s) = \sum_{n=1}^{\infty} 1/n^s \) for \( s > 1 \);

\[
(1.2) \quad \sum_{n \leq x} \varphi^*(n) = \frac{1}{2} ax^2 + O(x \log^2 x),
\]

where

\[
(1.3) \quad \alpha = \prod_p \left( 1 - \frac{1}{p(p+1)} \right),
\]

the product being extended over all primes \( p \).

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The object of the present paper is to improve the order estimates of the error terms in the asymptotic formulae (1.1) and (1.2). In fact, we prove that

\begin{equation}
\sum_{n \leq x} \sigma^*(n) = \frac{\pi^2 x^2}{12 \zeta(3)} + O(x \log^{5/3} x),
\end{equation}

\begin{equation}
\sum_{n \leq x} \varphi^*(n) = \frac{1}{2} x^{3/2} + O(x \log^{5/3} x(\log \log x)^{4/3}),
\end{equation}

where \( \alpha \) is the constant given by (1.3).

2. Prerequisites. In this section, we prove some lemmas which are needed in the present discussion. Let \( \mu(n) \) denote the Möbius function, \( \varphi(n) \) denote the Euler totient function, \( \sigma(n) \) denote the sum of the divisors of \( n \) and \( \gamma(n) \) denote the core (the maximal square-free divisor) of \( n \). Let \( a(n) \) be the arithmetical function defined by \( a(1)=1 \) and \( a(n) = \prod_{i=1}^{r} (\alpha_i - 1) \) if \( n = \prod_{i=1}^{r} p_i^{\alpha_i} \).

It is known (cf. [1, Corollary 2.2.1]) that \( \varphi^*(n) \) is multiplicative, and so we have

\begin{equation}
\varphi^*(n) = n \prod_{i=1}^{r} \left( 1 - \frac{1}{p_i^{\alpha_i}} \right),
\end{equation}

if \( n = \prod_{i=1}^{r} p_i^{\alpha_i} \).

We use the following best-known estimates due to A. Walfisz [4] concerning the averages of the functions \( \sigma(n) \) and \( \varphi(n) \):

**Lemma 2.1** (cf. [4, Satz 4, p. 99]). For \( x \geq 2 \),

\begin{equation}
\sum_{n \leq x} \sigma(n) = \frac{\pi^2 x^2}{12} + O(x \log^{2/3} x).
\end{equation}

**Lemma 2.2** (cf. [4, Satz 1, p. 144]). For \( x \geq 3 \),

\begin{equation}
\sum_{n \leq x} \varphi(n) = \frac{3x^2}{\pi^2} + O(x \log^{2/3} x(\log \log x)^{4/3}).
\end{equation}

**Lemma 2.3.** \( \sigma^*(n) = \sum_{d^2|n} \mu(d) \sigma(d) \).

**Proof.** Since \( \mu(n) \) and \( \sigma(n) \) are multiplicative, it follows (cf. [3, Lemma 2.4]) that \( \sum_{d^2|n} \mu(d) \sigma(d) \) is a multiplicative function of \( n \). Also, \( \sigma^*(n) \) is multiplicative (cf. [1, Lemma 6.1]). Hence, it is enough if we verify the identity for \( n=p^a \), a prime power. We have

\[
\sum_{d^2|p} \mu(d) \sigma(d) = \mu(1)\sigma(p) = 1 + p = \sigma^*(p),
\]
and for \( \alpha \geq 2 \),
\[
\sum_{\delta \mid n^\alpha} \mu(d) d \sigma(\delta) = \mu(1) \sigma(p^\alpha) + \mu(p) p \sigma(p^{\alpha-2})
\]
\[
= (1 + p + \cdots + p^\alpha) - p(1 + p + \cdots + p^{\alpha-2})
\]
\[
= 1 + p^\alpha = \sigma^*(p^\alpha).
\]
Hence the lemma follows.

**Lemma 2.4.** \( \varphi^*(n) = \sum_{\delta \mid n} (\varphi(d)a(\delta) \varphi(\delta) \gamma(\delta)) / \delta \).

**Proof.** Since \( a(n), \gamma(n), \varphi(n) \) and \( \varphi^*(n) \) are multiplicative, it is enough, if we verify the identity for \( n = p^\alpha \), a prime power. We have
\[
\sum_{\delta \mid n^\alpha} \varphi(d)a(\delta) \varphi(\delta) \gamma(\delta) / \delta
\]
\[
= \varphi(1)a(p^\alpha)(p - 1) + \varphi(p)a(p^{\alpha-1})(p - 1) + \cdots
\]
\[
+ \varphi(p^{\alpha-2})a(p^\alpha)(p - 1) + \varphi(p^\alpha)a(1)
\]
\[
= (p - 1)((\alpha - 1) + p(1 - 1/p)(\alpha - 2) + \cdots + p^{\alpha-2}(1 - 1/p))
\]
\[
+ \varphi(p^\alpha)
\]
\[
= (p - 1)(\alpha - 1) + (p - 1)^2
\]
\[
\times ((\alpha - 2) + p(\alpha - 3) + \cdots + p^{\alpha-3}) + p^\alpha(1 - 1/p)
\]
\[
= (p - 1)(\alpha - 1) + (p - 1)^2
\]
\[
\times \left( \frac{\alpha(p^{\alpha-2} - 1)}{p - 1} + 1 - \frac{1 - \alpha p^{\alpha-1} + (\alpha - 1)p^\alpha}{p(p - 1)^2} \right)
\]
\[
+ (p^\alpha - p^{\alpha-1})
\]
\[
= (p - 1)(\alpha - 1) + (p^\alpha + p + \alpha - \alpha p - 2) + (p^\alpha - p^{\alpha-1})
\]
\[
= p^\alpha - 1 = \varphi^*(p^\alpha).
\]
Hence the lemma follows.

**Lemma 2.5.** \( A(x) = \sum_{n \leq x} a(n)\gamma(n) = O(x) \).

**Proof.** We have \( 1 / \zeta(s) = \prod_p (1 - 1/p^s) \) for \( s > 1 \) (cf. [2, Theorem 280]). Hence
\[
\frac{1}{\zeta(2s - 1)} \sum_{n=1}^\infty a(n)\gamma(n) / n^s
\]
\[
= \prod_p \left( \left(1 - \frac{1}{p^{2s}}\right) \left(1 + \frac{p}{p^{2s}} + \frac{2p}{p^{3s}} + \frac{3p}{p^{4s}} + \cdots\right)\right)
\]
\[
= \prod_p \left( 1 + \frac{2p}{p^{3s}} + \frac{3p - p^2}{p^{4s}} + \frac{4p - 2p^2}{p^{5s}} + \cdots\right)
\]
\[
= \sum_{n=1}^\infty b(n) / n^s, \quad \text{say.}
\]
Hence
\[ \sum_{n=1}^{\infty} \frac{a(n)\gamma(n)}{n^s} = \zeta(2s - 1) \sum_{n=1}^{\infty} \frac{b(n)}{n^s} = \sum_{n=1}^{\infty} \frac{n}{n^s} \sum_{n=1}^{\infty} \frac{b(n)}{n^s}, \]
so that
\[ (2.5) \quad a(n)\gamma(n) = \sum_{d|n} db(\delta). \]

Also, by (2.4) we see that the abscissa of absolute convergence \( \beta \) of \( \sum_{n=1}^{\infty} (b(n)/n^s) \) is \( \beta = \frac{3}{4} \). Hence, we have
\[ (2.6) \quad \sum_{n \leq x} b(n) = O(x^{3/4+\epsilon}), \quad \text{for every } \epsilon > 0. \]

Hence by (2.5) and (2.6), we have
\[ \sum_{n \leq x} a(n)\gamma(n) = \sum_{d|n} db(\delta) = \sum_{d \leq x^{1/2}} d \sum_{d \leq x/d^3} b(\delta) \]
\[ = O \left( \sum_{d \leq x^{1/2}} \frac{x^{3/4+\epsilon}}{d^2} \right) \]
\[ = O \left( x^{3/4+\epsilon} \sum_{d \leq x^{1/2}} \frac{1}{d^{1/2+2\epsilon}} \right) \]
\[ = O(x^{3/4+\epsilon} (x^{1/2}/2^{2\epsilon})^{1/2-2\epsilon}) = O(x). \]

**Lemma 2.6.** \( \sum_{n=1}^{\infty} (a(n)\varphi(n)\gamma(n)/n^s) = \zeta(2)\alpha \), where \( \alpha \) is given by (1.3).

**Proof.** Applying Lemma 2.5 and partial summation, it is easy to show that \( \sum_{n \leq x} (a(n)\gamma(n)/n^s) = O(1) \). Now since \( \varphi(n) \leq n \), it follows that the series \( \sum_{n=1}^{\infty} (a(n)\varphi(n)\gamma(n)/n^s) \) is absolutely convergent. Further, the general term of the series is multiplicative. Hence the series can be expanded into an infinite product of Euler type (cf. [2, Theorem 286]), so that we have
\[ \sum_{n=1}^{\infty} a(n)\varphi(n)\gamma(n)/n^s = \prod_p \left( 1 + \sum_{i=2}^{\infty} \frac{(i-1)(p-1)}{p^{2i}} \right) \]
\[ = \prod_p \left( 1 + \frac{p-1}{p^2(1-p^{-2})^2} \right) = \frac{\prod_p (1 - 2/p^2 + 1/p^3)}{\prod_p (1 - 1/p^3)^2} \]
\[ = \zeta^2(2) \prod_p \left( 1 - \frac{2}{p^2} + \frac{1}{p^3} \right) = \zeta(2) \prod_p \left( 1 - \frac{1}{p(p+1)} \right). \]

Hence the lemma follows.

**Lemma 2.7.** For \( x \geq 2 \),
\[ (2.7) \quad \sum_{n > x} \frac{a(n)\gamma(n)}{n^2} = O \left( \frac{1}{x} \right). \]
and

(2.8) \[ \sum_{n \leq x} \frac{a(n)\gamma(n)}{n} = O(\log x). \]

**Proof.** Both (2.7) and (2.8) follow by partial summation and Lemma 2.5.

3. Proofs of (1.4) and (1.5). By Lemmas 2.3 and 2.1, we have

\[
\sum_{n \leq x} \varphi^*(n) = \sum_{d \leq \sqrt{x}} \mu(d) \varphi(d) \sigma(d) = \sum_{d \leq \sqrt{x}} \mu(d) \sum_{\delta \leq d/\delta} \sigma(\delta)
\]

\[
= \sum_{d \leq \sqrt{x}} \mu(d) \left( \frac{\pi^2}{12} \left( \frac{d^2}{x} \right)^2 + O \left( \frac{d^2}{x} \log \frac{x}{d^2} \right) \right)
\]

\[
= \frac{\pi^2 x^2}{12} \sum_{n \leq x^{1/2}} \frac{\mu(n)}{n^3} + O \left( \log \frac{x}{3} \sum_{n \leq x^{1/2}} \frac{1}{n} \right)
\]

\[
= \frac{\pi^2 x^2}{12} \sum_{n=1}^\infty \frac{\mu(n)}{n^3} + O \left( x^2 \sum_{n > x^{1/2}} \frac{1}{n^3} \right) + O(x \log^{5/3} x)
\]

\[
= \frac{\pi^2 x^2}{12} + O(x) + O(x \log^{5/3} x),
\]

since \( \sum_{n=1}^\infty \frac{\mu(n)}{n^3} = \frac{1}{\zeta(3)} \) (cf. [2, Theorem 287]). Hence (1.4) follows.

By Lemmas 2.4 and 2.2, we have

\[
\sum_{n \leq x} \varphi^*(n) = \sum_{d \leq \sqrt{x}} \varphi(d) \sigma(d) \gamma(d)
\]

\[
= \sum_{d \leq \sqrt{x}} \varphi(d) \gamma(d) \sum_{\delta \leq d/\delta} \varphi(\delta)
\]

\[
= \sum_{\delta \leq x} \varphi(\delta) \gamma(\delta) \left( \frac{3x^2}{\pi^2 \delta^2} + O \left( \frac{x}{\delta} \log^{2/3} \left( \frac{x}{\delta} \right) \log \log \left( \frac{x}{\delta} \right)^{4/3} \right) \right)
\]

\[
= \frac{3x^2}{\pi^2} \sum_{n \leq x} \frac{a(n)\varphi(n)\gamma(n)}{n^3}
\]

\[
+ O \left( x \log^{2/3} x (\log \log x)^{4/3} \sum_{n \leq x} \frac{a(n)\varphi(n)\gamma(n)}{n^2} \right)
\]

(3.1) \[
= \frac{x^2}{2\zeta(2)} \sum_{n=1}^\infty \frac{a(n)\varphi(n)\gamma(n)}{n^3} + O \left( x^2 \sum_{n > x} \frac{a(n)\varphi(n)}{n^2} \right)
\]

\[
+ O \left( x \log^{2/3} x (\log \log x)^{4/3} \sum_{n \leq x} \frac{a(n)\varphi(n)}{n} \right),
\]

since \( \varphi(n) \leq n \). Now, the first O-term in (3.1) is \( O(x^2 \cdot 1/x) = O(x) \), by
(2.7) and the second $O$-term in (3.1) is $O(x \log^{5/3} x (\log \log x)^{1/3})$, by (2.8). Hence (1.5) follows by Lemma 2.6.

REFERENCES


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