PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 41, Number 1, November 1973

# EXPLICIT CONDITIONS FOR THE FACTORIZATION OF $n$TH ORDER LINEAR DIFFERENTIAL OPERATORS

ANTON ZETTL

ABSTRACT. For any integer $k$ with $1 \leqq k \leqq n$ sufficient conditions on the coefficients $p_i$ are given for the factorization of certain classes of operators $Ly = p_n y^{(n)} + p_{n-1} y^{(n-1)} + \cdots + p_0 y$ into a product $L = PQ$ where $P$ and $Q$ are operators of the same type of orders $n-k$ and $k$, respectively. A special case yields that if $(-1)^k p_0 \geqq 0$ then $y^n + p_0 y$ is factorable into a product of two regular differential operators of orders $n-k$ and $k$.

1. **Introduction.** We consider the classical $n$th order regular linear differential operator

$$Ly = p_n y^{(n)} + p_{n-1} y^{(n-1)} + \cdots + p_0 y \tag{1.1}$$

where $p_i$ for $i=0, 1, \cdots, n$ are continuous real valued functions with $p_n(t) \neq 0$ on some interval $[a, b)$ for $-\infty < a < b \leqq \infty$.

As a consequence of our main theorem we obtain the following factorization results: Suppose $p_n(t) \equiv 1$, $p_i \equiv 0$ for $i=1, 2, \cdots, n-3$ and $p_{n-2} \leqq 0$. (a) If $p_0 \geqq 0$, then $L$ has a right factor of any even order, i.e., for any even positive integer $k < n$ there exist operators $P$ and $Q$ of type (1.1) and orders $n-k$ and $k$, respectively, such that $L = PQ$. (b) If $p_0 \leqq 0$, then $L$ has a right factor of any odd order.

Right factors $Q$ of order $n-1$ are obtained under a much weaker hypothesis. For the case when the order of $Q$ is 1 stronger results are well known. Some extensions of these results are indicated as well as a generalization to a quasi-differential operator. Also a couple of applications to boundary value problems are given.

According to a well-known result of Pólya [12] the operator $L$ has a factorization into "products" of first order operators

$$Ly = r_n(r_{n-1} \cdots (r_1(r_0 y)')' \cdots)' \tag{1.2}$$

if and only if the equation $Ly=0$ has a fundamental set of solutions

Received by the editors October 6, 1972 and, in revised form, February 20, 1973.
*AMS (MOS) subject classifications* (1970). Primary 34A30, 34A05; Secondary 34A01.
*Key words and phrases.* Ordinary differential equations, factoring differential operators, Wronskians, property "W", linear homogeneous differential equations.

$y_1, \cdots, y_n$ such that

$$W_k > 0 \quad \text{for } k = 1, \cdots, n-1 \tag{1.3}$$

where $W_1 = y_1$ and $W_k = \det_{i,j=1,\cdots,k}[y_j^{(i-1)}]$ for $k=2, \cdots, n-1$. For a short and elegant proof of this factorization see **[13]**. A factorization of type (1.2) on an interval $(a, b)$ is known to be equivalent to disconjugacy on $(a, b)$. The problem of finding explicit conditions on the coefficients $p_i$ which yield a factorization of type (1.2) or, equivalently, assure that $L$ is disconjugate has received considerable attention. For some recent papers see **[1]**, **[6]**, **[11]**, **[14]**, **[15]**.

In **[17]** it is shown that an operator $L$ of type (1.1) has a factorization

$$L = PQ \tag{1.4}$$

where $P$ and $Q$ are of type (1.1) of orders $n-k$ and $k$, respectively, if and only if there exist $k$ linearly independent solutions of $Ly=0$ whose Wronskian $W_k$ satisfies

$$W_k > 0. \tag{1.5}$$

Not much seems to be known about conditions which yield factorizations of type (1.4). Some conditions—involving the Lagrange bilinear form for solutions—which imply a factorization of some $2n$th order operators as products of two $n$th order ones were obtained by Rellich and Heinz in **[4]**. Direct conditions on the coefficients which yield factorizations of types (1.2) and (1.4) are obtained in **[16]** for the case $n=3$.

2. We use the notation $X \geqq 0$ for a matrix or vector $X$ to mean that each component of $X$ is nonnegative. Similarly $X > 0$ means each component is strictly positive.

Our development is based on the following two lemmas. The first one is a very useful result due to Mikusinski **[9]**. The second one is the result stated above from **[17]**.

LEMMA 1. *Let $y_i' = \sum_{j=1}^m F_{ij} y_j$ for $i=1, \cdots, m$ be a system of differential equations with real valued continuous coefficients which are nonnegative for $i \neq j$ on $[a, b)$. If $Y = (y_i)$ for $i=1, \cdots, m$ is a solution vector satisfying $Y(a) \geqq 0$, then $Y(t) \geqq 0$ for $t$ in $[a, b)$. Moreover if some component $y_i$ is positive at a point $c$ in $[a, b)$, then $y_i(t)$ is positive for $t > c$.*

LEMMA 2. *A necessary and sufficient condition that a differential operator $L$ of type* (1.1) *has a factorization* (1.4) *where $P$ and $Q$ are operators of type* (1.1) *of orders $n-k$ and $k$ respectively is that there exist $k$ linearly independent solutions of $Ly=0$ whose Wronskian $W_k$ is positive.*

For convenience of notation we denote the solution space of $Ly=0$ by $S$. In the rest of the paper we assume, for convenience, that $p_n(t) \equiv 1$.

As a consequence of Lemma 1 and the classical vector matrix representation of the equation $Ly=0$ we obtain

THEOREM 1. *Suppose* $p_i \leqq 0$ *for* $i=0, 1, \cdots, n-2$. *If* $y \in S$ *with* $y^{(i)}(a) \geqq 0$ *for* $i=0, 1, \cdots, n-1$ *and* $y^{(r)}(a)>0$, *then* $y^{(i)}(t) \geqq 0$ *for* $i=0, \cdots, n-1$ *and* $t>a$ *and* $y^{(p)}(t)>0$ *for* $p=0, \cdots, r$ *and* $t>a$.

Using positive initial conditions at $a$ to determine a solution of $Ly=0$, then as a consequence of Theorem 1 and Lemma 2 we obtain

COROLLARY 1. *If* $p_i \leqq 0$ *for* $i=0, 1, \cdots, n-2$; *then the operator* $L$ *can be factored*: $L=PQ$ *where* $P$ *is of order* $n-1$ *and* $Q$ *is of order* 1.

Theorem 1 is stated only for the sake of completeness since the result is known—although many authors use a sign condition on $p_{n-1}$.

The conclusion of Corollary 1 still holds if the signs of the coefficients alternate—see [**2**, p. 508].

THEOREM 2. *If* $(-1)^{n-1} p_i \leqq 0$ *for* $i=0, 1, \cdots, n-2$ *then there exists* $y \in S$ *with* $y(t)>0$ *for* $t>a$.

Let $z_1, \cdots, z_n$ be solutions of $Ly=0$ determined by the initial conditions

$$z_j^{(i-1)}(a) = \delta_{ij} \quad \text{for } i, j = 1, \cdots, n. \tag{2.2}$$

Our main result is:

THEOREM 3. *Suppose* $p_i \equiv 0$ *for* $i=1, 2, \cdots, n-3$ *and* $p_{n-2} \leqq 0$. *Let* $W_k$ *denote any kth order Wronskian* $W(z_i, z_{i+1}, \cdots, z_{i+k-1})$ *for* $i=1, 2, \cdots, n-k+1$, $k=1, \cdots, n-1$. *If* $p_0 \geqq 0$ *and* $k$ *is even, then* $W_k>0$ *on* $(a, b)$. *If* $p_0 \leqq 0$ *and* $k$ *is odd, then* $W_k>0$ *on* $(a, b)$.

PROOF. Let $y_1, \cdots, y_k$ be any solutions of $Ly=0$. For integers $i_j \leqq n$, $j=1, \cdots, k$, we define

$$D(i_1, i_2, \cdots, i_k) = \det \begin{vmatrix} y_1^{i_1}, & y_2^{i_1}, & \cdots, & y_k^{i_1} \\ y_1^{i_2}, & y_2^{i_2}, & \cdots, & y_k^{i_2} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ y_1^{i_k}, & y_2^{i_k}, & \cdots, & y_k^{i_k} \end{vmatrix}.$$

Note that $D(i_1, \cdots, i_k)=0$ if any pair of $i_j$'s are equal and that the $i_j$'s can always be put in increasing order by a change in sign, if necessary. Also note that $D(i_1, i_2, \cdots, i_{k-1}, n)$ can be expressed in terms of determinants involving only derivatives of orders less than $n$ by replacing $y_i^{(n)}$ by $-p_{n-1} y_i^{(n-1)} - \cdots - p_0 y_i$.

If $i_k < n-1$ observe that

$$D'(i_1, i_2, \cdots, i_k) = D(i_1+1, i_2, \cdots, i_k) + D(i_1, i_2+1, i_3, \cdots, i_k) + \cdots + D(i_1, i_2, \cdots, i_k+1).$$

If $i_k = n-1$, then

$$\begin{aligned} D'(i_1, \cdots, i_k) = {} & D(i_1+1, i_2, \cdots, i_k) + D(i_1, i_2+1, i_3, \cdots, i_k) \\ & + \cdots + D(i_1, i_2, \cdots, i_{k-1}+1, i_k) \\ & + D(i_1, i_2, \cdots, i_{k-1}, i_k+1 = n) \end{aligned}$$

and

$$\begin{aligned} D(i_1, i_2, \cdots, i_{k-1}, n) = {} & -p_0 D(i_1, i_2, \cdots, i_{k-1}, 0) - p_1 D(i_1, i_2, \cdots, i_{k-1}, 1) \\ & - p_2 D(i_1, i_2, \cdots, i_{k-1}, 2) - \cdots \\ & - p_{n-1} D(i_1, i_2, \cdots, i_{k-1}, n-1). \end{aligned}$$

From this it follows that the set of $D(i_1, i_2, \cdots, i_k)$ for $i_1 < i_2 < \cdots < i_k$, $i_j = 0, 1, \cdots, n-1, j = 1, \cdots, k$, are solutions of a system of differential equations $Y' = FY$ where $Y$ is a vector of order $\binom{n}{k}$ with components $D(i_1, i_2, \cdots, i_k)$ and $F$ is an $\binom{n}{k}$ by $\binom{n}{k}$ matrix whose entries consist of 0's and 1's and $(-1)^k p_0$, $+p_1$, $-p_1$, $+p_2$, $-p_2, \cdots, +p_{n-3}$, $-p_{n-3}$, $-p_{n-2}$, $-p_{n-1}$. The components of $Y$ are ordered as follows:

$Y =$ transpose of

$$\begin{aligned} & [D(0, 1, \cdots, k-1), D(0, 1, \cdots, k-2, k), \\ & D(0, 1, \cdots, k-2, k+1), \cdots, D(0, 1, \cdots, k-2, n-1), \\ & D(0, 1, \cdots, k-3, k-1, k), \\ & D(0, 1, \cdots, k-3, k-1, k+1), \cdots, \\ & \qquad\qquad D(0, 1, \cdots, k-3, k-1, n-1) \cdots]. \end{aligned}$$

Note that $-p_{n-1}$ is on the diagonal of $F$—hence no sign condition is needed for $p_{n-1}$ in order to use Lemma 1.

Therefore by Lemma 1, all the components of $Y$ are nonnegative on $[a, b)$ if $Y(a) \geqq 0$.

For any $i = 1, 2, \cdots, n-k+1$ let $y_1 = z_i$, $y_2 = z_{i+1}, \cdots, y_k = z_{i+k-1}$ where the $z_j$'s are the solutions determined by the initial conditions (2.2).

For such a choice of $y_1, y_2, \cdots, y_k$ the initial vector $Y(a) \geqq 0$ and the component $D(i-1, i, i+1, \cdots, i+k-2)$ has the value 1 at $a$. The first component of $Y$, namely $W_k$, is nondecreasing on $[a, b)$ since $W_k' = D(0, 1, \cdots, k-2, k-1) = D(0, 1, \cdots, k-2, k) \geqq 0$. Hence $W_k$ is positive on $(a, b)$ since $W_k$ identically zero on some open interval $(a, t_0)$ would imply that all the components of $Y$ are identically zero on $(a, t_0)$. But the component $D(i-1, i, \cdots, i+k-2)$ cannot be zero in $(a, t_0)$ since it is positive at $a$ and continuous. This completes the proof of Theorem 3.

The factorization result mentioned in the introduction is obtained by taking $i=1$ in the above argument and noting that $W(z_1, z_2, \cdots, z_k)(a)=1$ to get $W(z_1, z_2, \cdots, z_k)(t)>0$ for $t\geqq a$. The rest follows from Lemma 2.

We remark that, as shown in [**17**], the right factor $Q$ can be taken as $Qy=W(y_1, y_2, \cdots, y_k, y)$ where $y_1, y_2, \cdots, y_k$ is any set of solutions of $Ly=0$ satisfying $W_k=W(y_1, \cdots, y_k)>0$.

For the case $k=n-1$, we obtain a stronger result:

THEOREM 4. *Suppose* $(-1)^{n+1-j}p_j\geqq 0$ *for* $j=0, 1, \cdots, n-2$. *Then there exist* $y_1, y_2, \cdots, y_{n-1}\in S$ *such that* $W_{n-1}=W(y_1, y_2, \cdots, y_{n-1})>0$.

PROOF. The proof is similar to that of Theorem 3, therefore we merely outline it here.

Let solutions $y_i$ of $Ly=0$ be determined by initial conditions $y_i^{j-1}(a)=\delta_{ij}$ for $i=1, \cdots, n-1$ and $j=1, \cdots, n$. Define $D(i_1, i_2, \cdots, i_{n-1})$ as in the proof of Theorem 3. As in the proof of Theorem 3 we then show that the column vector

$$\begin{aligned} Y = [&D(0, 1, 2, \cdots, n-2), D(0, 1, 2, \cdots, n-3, n-1), \\ &D(0, 1, 2, \cdots, n-4, n-2, n-1), \cdots, \\ &D(0, 2, 3, \cdots, n-1), D(1, 2, 3, \cdots, n-1)] \end{aligned}$$

satisfies the differential system $Y'=FY$ where $F$ is the matrix having 1's on the super diagonal, $-p_{n-1}$ on the diagonal except for the (1, 1) and $(n, n)$ position, 0's elsewhere except for the first column which is, from top to bottom, $[0, -p_{n-2}, +p_{n-3}, \cdots, (-1)^n p_1, (-1)^{n+1}p_0]$. Noting that $Y(a)=[1, 0, \cdots, 0]$, we conclude, by Lemma 1, that

$$\begin{aligned} y_1(t) &= D(0, 1, 2, \cdots, n-2)(t) \\ &= W_{n-1}(t) = W(y_1, y_2, \cdots, y_{n-1})(t) > 0 \quad \text{for } t \geqq a. \end{aligned}$$

As an immediate consequence of Theorem 4 and Lemma 2 we obtain

COROLLARY 2. *Under the hypothesis of Theorem* 4, *L has a right factor* $Q$ *of order* $n-1$.

3. Here we indicate some extensions of these results and give a couple of applications.

REMARK 1. This is a result obtained by Kim in [**5**]. Suppose $u$, $v$ are solutions of $Ly=0$ satisfying $W_2=W(u, v)=uv'-vu'>0$ on $[a, b)$. By Lemma 2 we have a factorization $L=PQ$ where $Q$ has order 2 and by the remark following the proof of Theorem 3 we can take $u$, $v$ to be a fundamental set of solutions of $Qy=0$. Therefore we can conclude such things as:

(i) Neither $u$ nor $v$ can have a double zero on $[a, b)$.
(ii) $u^j$, $v^j$ have no common zero on $[a, b)$ for $j=0, 1$.
(iii) Between any two zeros of one there is a zero of the other.

REMARK 2. If $p_i \in C^i_{[a,b)}$ for $i=0, 1, \cdots, n$ then the classical adjoint operator $L^+$, defined by

$$L^+y = (-1)^n(p_n y)^{(n)} + (-1)^{(n-1)}(p_{n-y})^{(n-1)} + \cdots + p_0 y$$

can be put into the form (1.1). By applying the above factorization results to the adjoint operator $L^+$ and using the fact that $L=PQ$ if and only if $L^+=Q^+P^+$—see **[10]**—additional sufficient conditions for factorization can be obtained. We illustrate with an example: Consider the fourth order operators $L$ and $L^+$ defined by

$$\begin{aligned} Ly &= y^{(4)} + p_3 y^{(3)} + p_2 y^{(2)} + p_1 y^{(1)} + p_0 y, \\ L^+y &= y^{(4)} - (p_3 y)^{(3)} + (p_2 y)^{(2)} - (p_1 y)' + p_0 y \\ &= y^{(4)} - p_3 y''' + [p_2 - 3p_3']y'' \\ &\quad + [2p_2' - p_1 - 3p_3'']y' + [p_0 - p_1' + p_2'' - p_3''']. \end{aligned}$$

Applying our factorization theorems we have

COROLLARY 3. *If $p_2-3p_3'\leqq 0$, $2p_2'-p_1-3p_3''\leqq 0$, $p_0-p_1'+p_2''-p_3'''\leqq 0$, then $L^+=P_3Q_1$ where $P_3$, $Q_1$ are operators of type* (1.1) *of orders* 3 *and* 1, *respectively. Hence $L=Q_1^+P_3^+$.*

COROLLARY 4. *If $p_0-p_1'+p_2''-p_3'''\leqq 0$, $2p_2'-p_1-3p_3''\geqq 0$ and $p_2-3p_3'\leqq 0$, then $L=Q_1^+P_3^+$ where $Q_1$, $P_3$ are of orders* 1 *and* 3, *respectively.*

COROLLARY 5. *If $p_0-p_1'+p_2''-p_3'''\geqq 0$ and $2p_2'-p_1-3p_3''=0$ and $p_2-3p_3'\leqq 0$, then $L=P_2Q_2$ and $L^+=Q_2^+P_2^+$ where $P_2$, $Q_2$ are operators of type* (1.1) *of order* 2.

COROLLARY 6. *If $p_0-p_1'+p_2''-p_3'''\leqq 0$, $2p_2'-p_1-3p_3''\geqq 0$, $p_2-3p_3'\leqq 0$, then there exist operators $P_1$ and $Q_3$ of type* (1.1) *of orders* 1 *and* 3, *respectively such that $L^+=P_1Q_3$ and hence $L=Q_3^+P_1^+$.*

As another application of some of these factorizations we list

THEOREM 5. *Under the hypothesis of Theorem* 4, *the boundary value problem*

$$Ly = 0, \qquad y(\alpha) = 0, y(\beta) = y'(\beta) = \cdots = y^{(n-2)}(\beta) = 0$$

*for any $\alpha$, $\beta$ in $[a, b)$ with $\alpha<\beta$ has no nontrivial solution.*

PROOF. Suppose $y$ is a nontrivial solution. Let $c$ be the first point to the left of $\beta$ such that $y^{(i)}(c)\neq 0$ for $i=0, \cdots, n-2$ and $y^{(n-1)}(c)=0$.

Determine solutions $y_1, y_2, \cdots, y_{n-2}$ by the initial conditions:

$$y_1(c) = y(c), \quad y_1'(c) = 0, \quad \cdots, \quad y_1^{(n-1)}(c) = 0$$

$$y_2(c) = 0, \quad y_2'(c) = y'(c), \quad y_2''(c) = 0, \quad \cdots, \quad y_2^{(n-1)}(c) = 0$$

$$\vdots$$

$$y_{n-2}(c) = 0, \quad \cdots, \quad y_{n-2}^{(n-4)}(c) = 0, \quad y_{n-2}^{(n-3)}(c) = y^{(n-3)}(c),$$

$$y_{n-2}^{(n-2)}(c) = 0, \quad y_{n-2}^{(n-1)}(c)=0.$$

Define $D(i_1, i_2, \cdots, i_{n-1})$ as in the proof of Theorem 3 using $y=y_{n-1}$ and note that $D(0, 1, 2, \cdots, n-2)(c)=W(y_1, y_2, \cdots, y_{n-2}, y)(c)=y(c)y'(c)\cdots y^{(n-2)}(c)\neq 0$ and all other $D(i_1, i_2, \cdots, i_{n-1})$ are zero at $c$.

Repeated applications of the mean value theorem show that the signs of $y(c), y'(c), y''(c), \cdots, y^{(n-2)}(c)$ alternate. Here we are using the fact that $y^{(n-1)}(\beta)\neq 0$ and that $c$ is the first point to the left of $\beta$ such that $y^{(i)}(c)\neq 0$ for $i=0, \cdots, n-2$ and $y^{(n-1)}(c)=0$. By replacing $y$ with $-y$, if necessary, we can get the product $y(c)y'(c)\cdots y^{(n-2)}(c)$ positive. Proceeding as in the proof of Theorem 4 we get to the conclusion

$$W(t) = W(y_1, y_2, \cdots, y_{n-2}, y)(t) > 0 \quad \text{for } t > c.$$

But this contradicts $W(\beta)=0$.

We list here a couple of illustrations of Theorem 3.

THEOREM 6. *Under the hypothesis of Theorem* 3, *if $y$ is a nontrivial solution of $Ly=0$ on $[a, b)$ which has a zero of order $k$ at $a$ and a zero of order $n-k$ at $c$, $a<c<b$ then $n-k$ is even if $p_0\geqq 0$ and $n-k$ is odd if $p_0\leqq 0$.*

PROOF. Such a solution $y$ can be expressed as $y=\alpha_{k+1}z_{k+1}+\cdots+\alpha_n z_n$. A zero of order $n-k$ at $c>a$ would imply $W(z_{k+1}, \cdots, z_n)(c)=0$. A. Ju. Levin [7], [8] obtained, by different methods, this result for the operator $y^n+p_0y$.

A consequence of Theorem 6 is that, under the hypothesis of Theorem 3, no nontrivial solution of $Ly=0$ can have zeros at $a$, $c$ with $a<c<b$ of combined order $>n$, because this would imply that two Wronskians of consecutive integral order are zero at $c$. But one of these has to be even and one odd.

If an operator is given in quasi-differential form—such as $(py'')''+qy$—one can sometimes get simpler conditions by using the techniques of proof above and appropriate "quasi-derivatives" than by "stringing out" the expression into the form (1.1) and then using Theorems 1, 2, 3, and 4.

As an illustration we consider the operator

$$(3.1)\qquad Ly=(py'')''+qy$$

where $p\in C^2[a,b)$, $q\in C[a,b)$ and $p>0$ on $[a,b)$.

THEOREM 7. (a) *If $q\leqq 0$, then there exists a positive solution of $Ly=0$.*

(b) *If $q\geqq 0$ and $(pq)'\geqq 0$ then there exist solutions $u$, $v$ of $Ly=0$ such that $W_2=W(u,v)>0$. Hence $L$ has a factorization into a product of two second order operators.*

PROOF. *Part* (a). The proof is similar to that of Theorem 1. The main modification is that the vector $Y$ used here is $Y=$column vector $[y,y',py'',(py'')']$ and the resulting matrix $F$ has components all zero except for $-q$ in the (4, 1) position, 1 in the (1, 2) and (3, 4) positions and $1/p$ in the (2, 3) position. The details are omitted.

*Part* (b). Determine solutions $u$, $v$ of $Ly=0$ by the initial conditions:

$$u(a)=1,\quad u'(a)=0,\quad u''(a)=0,\quad u'''(a)=0,$$
$$v(a)=0,\quad v'(a)=1,\quad v''(a)=0,\quad v'''(a)=0.$$

Let $z=W(u,v)=uv'-u'v$. We show that the column vector $Y=[z,pz',(pz')',(pz')'',(pz')''']$ satisfies a differential system $Y'=FY$: Note that

$$z'=\begin{vmatrix} u, & v \\ u'', & v''\end{vmatrix},$$

$$(pz')'=\begin{vmatrix} u', & v' \\ pu'', & pv''\end{vmatrix}+\begin{vmatrix} u, & v \\ (pu'')', & (pv'')'\end{vmatrix},$$

$$(pz')''=2\begin{vmatrix} u', & v' \\ (pu'')', & (pv'')'\end{vmatrix},$$

$$(pz')'''=2\begin{vmatrix} u'', & v'' \\ (pu'')', & (pv'')'\end{vmatrix}+2qz$$

where we substituted $-qu$ for $(pu'')''$ and $-qv$ for $(pv'')''$ in $[p(pz')''']'=4q(pz')+2(pq)'z$.

From these computations we see that $Y'=FY$ where $F$ is the matrix with components zero everywhere except for 1's in the (2, 3) and (3, 4) positions, $1/p$ in the (1, 2) and (4, 5) positions, $+2(pq)$ in the (5, 1) position and $+rq$ in the (5, 2) spot. The conclusion follows from Lemmas 1 and 2.

## References


1. P. Hartman, *Principal solutions of disconjugate nth order linear differential equations*, Amer. J. Math. **91** (1969), 306–362. MR **40** #450.

2. ———, *Ordinary differential equations*, Wiley, New York, 1964. MR **30** #1270.

3. ———, *On disconjugacy criteria*, Proc. Amer. Math. Soc. **24** (1970), 347–381. MR **40** #4535.

4. E. Heinz, *Halbbeschränktheit gewöhnlicher Differentialoperatoren höherer Ordnung*, Vorwort von B. L. van der Waerden, Math. Ann. **135** (1958), 1–49. MR **21** #743.

5. W. J. Kim, *Oscillatory properties of linear third order differential equations*, Proc. Amer. Math. Soc. **26** (1970), 286–293. MR **41** #8758.

6. A. Ju. Levin, *Nonoscillation of solutions of the equation* $x^{(n)}+P_1(t)x^{(n-1)}+\cdots+P_n(t)x=0$, Uspehi Mat. Nauk **24** (1969), no. 2, (146), 43–96=Russian Math. Surveys **24** (1969), no. 2, 43–100. MR **40** #7537.

7. ———, *Some problems bearing on the oscillation of solutions of linear differential equations*, Dokl. Akad. Nauk SSSR **148** (1963), 512–515=Soviet Math. Dokl. **4** (1963), 121–124. MR **26** #3972.

8. ———, *Distribution of the zeros of solutions of a linear differential equation*, Dokl. Akad. Nauk SSSR **156** (1964), 1281–1284=Soviet Math. Dokl. **5** (1964), 818–821. MR **29** #1378.

9. J. Mikusinski, *Sur l'equation* $x^{(n)}+A(t)x=0$, Ann. Polon. Math. **1** (1955), 207–221. MR **19**, 141.

10. K. S. Miller, *Linear differential equations in the real domain*, Norton, New York, 1963. MR **27** #5947.

11. Z. Nehari, *Disconjugate linear differential operators*, Trans. Amer. Math. Soc., **129** (1967), 500–516. MR **36** #2860.

12. G. Pólya, *On the mean-value theorem corresponding to a given linear homogeneous differential equation*, Trans. Amer. Math. Soc. **24** (1922), 312–324.

13. R. Ristroph, *Pólya's property* W *and factorization—A short proof*, Proc. Amer. Math. Soc. **31** (1972), 631–632.

14. D. Willett, *Generalized De La Vallee Poussin disconjugacy tests for linear differential equations*, Canad. Math. Bull. (to appear).

15. ———, *Disconjugacy tests for singular differential equations*, Siam J. Math. Anal. **2** (1971), 536—546.

16. A. Zettl, *Factorization and disconjugacy of third order differential equations*, Proc. Amer. Math. Soc. **31** (1972), 203–208.

17. ———, *Factorization of differential operators*, Proc. Amer. Math. Soc. **27** (1971), 425–426. MR **42** #7966.


Department of Mathematics, Northern Illinois University, DeKalb, Illinois 60115