NOTE ON A LIMIT-POINT CRITERION

IAN KNOWLES

Abstract. A sufficient condition is given for the formal differential operator \( \tau y(t) = (p(t)y'(t))' + q(t)y(t) \) defined on the interval \([a, b), b \leq \infty\), to be of limit-point type at \(b\); this generalizes a criterion of Ismagilov given for the case \(p(t)=1\) and \(b=\infty\).

We consider the linear second order differential operator \(\tau\) defined on the real half-open interval \([a, b), b \leq \infty\), by

\[
\tau y(t) = (p(t)y'(t))' + q(t)y(t)
\]

where \(p^{-1}(t)>0\) and \(q(t)\) are real-valued functions locally Lebesgue integrable on \([a, b)\). The operator \(\tau\) is said to be of limit-circle type at \(b\) if every solution \(f(t)\) of the differential equation \(\tau y(t)=0\) satisfies the condition

\[
\int_a^b |f^2(t)| \, dt < \infty.
\]

If this is not the case, then \(\tau\) is said to be of limit-point type at \(b\).

In [1], Ismagilov proved that if \(p(t)=1\) and \(q(t)\leq -q_n<0\) on disjoint intervals \(I_n \subset [a, \infty)\) of length \(\mu_n\) and \(\sum_{n=1}^{\infty} q_n^{1/2} \mu_n^2 = \infty\) then \(\tau\) is of limit-point type at \(\infty\). We present here the corresponding result for the general operator \(\tau\) defined on an arbitrary half-open subinterval of the real line.

**Theorem.** If there exist finite disjoint subintervals \(I_n, n=1, 2, \cdots\), of \([a, b)\) and corresponding numbers \(q_n\) and \(p_n\) such that \(q(t)\leq -q_n<0\) and \(p(t)\geq p_n>0\) on \(I_n\) and

\[
\sum_{n=1}^{\infty} p_n^{3/2} q_n^{1/2} \left( \int_{I_n} p^{-1}(s) \, ds \right)^3 = \infty
\]

then the operator \(\tau\) defined by equation (1) is of limit-point type at \(b\).

**Proof.** For each \(n=1, 2, \cdots\), let the interval \(I_n\) have endpoints \(a_n\) and \(b_n\), where \(a_n < b_n\), and let \(P(s) = \int_a^s p^{-1}(u) \, du\) for any \(s \in [a, b)\). We define the function \(h(s)\) on \(\bigcup_n I_n\) by

\[
h(s) = (P(b_n) - P(a_n))^{-2}(P(s) - P(a_n))^2(P(b_n) - P(s))^2
\]

Received by the editors January 30, 1973.


Key words and phrases. Differential operator, second order, limit-point type.
for $s \in I_n$. Then $h(a_n)=h(b_n)=h'(a_n)=h'(b_n)=0$ and, on each interval $I_n$,

$$p(s)(p(s)h'(s))' = 2(P(b_n) - P(a_n))^2[6P^2(s) - 6P(s)(P(b_n) + P(a_n)) + P^2(a_n) + 4P(a_n)P(b_n) + P^2(b_n)] \leq 2$$

because this is a quadratic function in the variable $P(s)$ which takes its maximum value at either endpoint of $I_n$. Also, after integrating by parts twice, it is easily seen that for any solution $f(s)$ of $\tau y(s)=0$,

$$\int_{I_n} f^2(s)(p(s)h'(s))' \, ds = 2\int_{I_n} h(s)[p(s)(f'(s))^2 - q(s)f^2(s)] \, ds.$$ 

(6) \hspace{1cm} \int_{I_n} f^2(s)(p(s)h'(s))' \, ds = 2\int_{I_n} h(s)[p(s)(f'(s))^2 - q(s)f^2(s)] \, ds.

Now let $u(s)$ and $v(s)$ be any two independent real-valued solutions of $\tau y(s)=0$ for which $p(uv' - u'v) = 1$. Then, on each interval $I_n$,

$$2(-q)^{1/2}p^{-1/2} = 2(-q)^{1/2}u(p^{1/2}v') - 2(-q)^{1/2}v(p^{1/2}u') \leq p[(u')^2 + (v')^2] - q[u^2 + v^2].$$

Hence

$$\int_a^b (u^2 + v^2) \, ds \geq \sum_{n=1}^\infty \int_{I_n} (u^2 + v^2) \, ds$$

$$\geq \sum_{n=1}^\infty \left[ \sup_{I_n} p(\text{ph'}) \right]^{-1} \int_{I_n} (u^2 + v^2)p(\text{ph'})' \, ds$$

$$\geq \sum_{n=1}^\infty p_n \left[ \sup_{I_n} p(\text{ph'}) \right]^{-1} \int_{I_n} (u^2 + v^2)(\text{ph'})' \, ds$$

$$= 2 \sum_{n=1}^\infty p_n \left[ \sup_{I_n} p(\text{ph'}) \right]^{-1} \int_{I_n} h[\text{p}((u')^2 + (v')^2) - q(u^2 + v^2)] \, ds \quad \text{from (6)}$$

$$\geq 4 \sum_{n=1}^\infty p_n \left[ \sup_{I_n} p(\text{ph'}) \right]^{-1} \int_{I_n} h(-q)^{1/2}p^{-1/2} \, ds \quad \text{from (7)}$$

$$\geq 4 \sum_{n=1}^\infty q_n^{1/2}P_n^{3/2} \left[ \sup_{I_n} p(\text{ph'}) \right]^{-1} \int_{I_n} h(s)p^{-1}(s) \, ds$$

$$\geq 2 \sum_{n=1}^\infty q_n^{1/2}P_n^{3/2} \int_{I_n} h(s)p^{-1}(s) \, ds \quad \text{using the property (5)}$$

$$= \frac{1}{15} \sum_{n=1}^\infty q_n^{1/2}P_n^{3/2} \left( \int_{I_n} p^{-1}(s) \, ds \right)^3 = \infty.$$

This completes the proof. Q.E.D.
At this stage, one might reasonably expect to improve this theorem by using another function $g(s)$ satisfying

$$g(a_n) = g(b_n) = g'(a_n) = g'(b_n) = 0, \quad n = 1, 2, \cdots,$$

in the proof, instead of the function $h(s)$ defined by (4). Unfortunately, no such improvement is possible. To see this, let

$$T_n(g) = \left[ \sup_{I_n} p(pg') \right]^{-1} \int_{I_n} g(s)p^{-1}(s) \, ds,$$

and substitute $g(s)$ for $h(s)$ in the preceding proof; the inequality (8) can now be rewritten in the form

$$\int_a^b (u^2 + v^2) \, ds \geq 4 \sum_{n=1}^\infty q_n^{1/2} p_n^{3/2} T_n(g),$$

which shows that $\tau$ is of limit-point type at $b$ if we have $\sum_{n=1}^\infty q_n^{1/2} p_n^{3/2} T_n(g) = \infty$. But,

$$T_n(g) = \frac{1}{2} \left[ \sup_{I_n} p(pg') \right]^{-1} \int_{I_n} (P(s) - P(a_n))^2 p^{-1}(s) g'(s) \, ds \leq \frac{1}{2} \int_{I_n} (P(s) - P(a_n))^2 p^{-1}(s) \, ds = \frac{1}{6} \left( \int_{I_n} p^{-1}(s) \, ds \right)^3,$$

and this criterion is therefore only a special case of the theorem. Thus, choosing another function $g(s)$ can at best only improve the constant appearing in the inequality (9).

REFERENCES


School of Mathematical Sciences, Flinders University of South Australia, Bedford Park, South Australia

Department of Mathematics, University of the Witwatersrand, Johannesburg, South Africa (Current address)