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GRONWALL'S INEQUALITY IN n INDEPENDENT VARIABLES

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ABSTRACT. The paper presents an extension of Gronwall's inequality to n independent variables. The inequality is established by solving a characteristic initial value problem by the Riemann method. Thus a Riemann function associated with a hyperbolic partial differential equation appears in the inequality. By using a representation of the Riemann function, the result is shown to coincide with an earlier result obtained by Walter using an entirely different approach.

1. **Introduction.** Gronwall's one-dimensional inequality [**1**], also known in a generalized form as Bellman's lemma [**2**], has been extended to several independent variables by different authors. For example, in [**3**] Conlan and Diaz obtained a generalization of Gronwall's inequality in n variables in order to prove uniqueness of solution of a nonlinear partial differential equation. In [**4**, p. 125] Walter gave a more natural extension of Gronwall's inequality in any number of variables by using the properties of monotone operators. Recently, by using the notion of a Riemann function, Snow [**5**], [**6**] obtained corresponding inequalities in two independent variables for scalar and vector functions. It turns out, as will be shown in this note, that Snow's technique in the scalar case can be employed to establish Gronwall's inequality in n independent variables which coincides with the result given in [**4**] when a representation of the Riemann function is used.

2. **The inequality.** Let Ω be an open bounded set in R^n and let a point $(x_1, \cdots, x_n)$ in Ω be denoted by x. Let x^0 and x $(x^0<x)$ be any two points in Ω and denote by D the parallelepiped defined by $x^0<\xi<x$ (that is, $x^0<\xi_i<x_i$, $1\leqq i\leqq n$).

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THEOREM. *Suppose $\phi(x)$, $a(x)$, and $b(x)\geqq 0$ are continuous functions in Ω. Let $v(\xi; x)$ be a solution of the characteristic initial value problem*

$$(1)\qquad \begin{aligned} &(-1)^n v_{\xi_1\cdots\xi_n}(\xi; x) - b(\xi)v(\xi; x) = 0 \quad \text{in } \Omega,\\ &v(\xi; x) = 1 \quad \text{on } \xi_i = x_i,\ i = 1, \cdots, n, \end{aligned}$$

and let D^+ be a connected subdomain of Ω containing x such that $v\geqq 0$ for all $\xi \in D^+$. If $D\subset D^+$ and

$$(2)\qquad \phi(x) \leqq a(x) + \int_{x^0}^{x} b(\xi)\phi(\xi)\, d\xi,$$

then

$$(3)\qquad \phi(x) \leqq a(x) + \int_{x^0}^{x} a(\xi)b(\xi)v(\xi; x)\, d\xi.$$

(Here $\int_{x^0}^{x}\cdots d\xi$ indicates the n-fold integral

$$\int_{x_1^0}^{x_1}\cdots\int_{x_n^0}^{x_n}\cdots d\xi_1\cdots d\xi_n.)$$

PROOF. Set

$$(4)\qquad u(x) = \int_{x^0}^{x} b(\xi)\phi(\xi)\, d\xi$$

so that

$$(5)\qquad D_1\cdots D_n u(x) = b(x)\phi(x),\quad D_i = \partial/\partial x_i,\qquad 1\leqq i\leqq n.$$

Since $b(x)\geqq 0$ in D, it follows from (2) and (4) that

$$(6)\qquad Lu \equiv D_1\cdots D_n u(x) - b(x)u(x) \leqq a(x)b(x) \quad\text{and}$$

$$(7)\qquad u(x) = 0 \quad \text{on } x_i = x_i^0,\ 1\leqq i\leqq n.$$

Furthermore, all pure mixed derivatives of u with respect to $x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n$ up to order $n-1$ vanish on $x_i = x_i^0$, $1\leqq i\leqq n$. If w is a function which is n times continuously differentiable in D, then

$$(8)\quad wLu - uMw = \sum_{k=1}^{n}(-1)^{k-1}D_k[(D_0D_1\cdots D_{k-1}w)(D_{k+1}\cdots D_nD_{n+1}u)]$$

where $Mw=(-1)^nD_1\cdots D_nw(x)-b(x)w(x)$ with $D_0\equiv D_{n+1}=I$ the identity. By integrating (8) over D, using ξ as variables of integration, and noting that u vanishes together with all its mixed derivatives up to order

$n-1$ on $\xi_k=x_k^0$, $1\leqq k\leqq n$, we then obtain

$$\int_D (wLu - uMw)\,d\xi = \sum_{k=1}^{n}(-1)^{k-1}\int_{\xi_k=x_k} (D_1\cdots D_{k-1}w)(D_{k+1}\cdots D_n u)\,d\xi' \tag{9}$$

where $d\xi'=d\xi_1\cdots d\xi_{k-1}d\xi_{k+1}\cdots d\xi_n$.

Now let w be chosen as the function v satisfying (1). Since $v=1$ on $\xi_k=x_k$, $1\leqq k\leqq n$, it follows that $D_1\cdots D_{k-1}v(\xi;x)=0$ on $\xi_k=x_k$ for $2\leqq k\leqq n$. Thus (9) becomes

$$\int_D v(\xi;x)Lu(\xi)\,d\xi = \int_{\xi_1=x_1} v(\xi;x)D_2\cdots D_n u(\xi)\,d\xi' = u(x). \tag{10}$$

By the continuity of v and by the fact that $v=1$ on $\xi=x$, there is a domain D^+ containing x on which $v\geqq 0$. Hence by multiplying (6) throughout by v and using (4) and (10), we obtain (3).

3. **A representation of** $v(\xi;x)$. We observe that the problem (1) defines precisely the so-called Riemann function for the operator L. The existence and regularity property of v can be deduced from [3] (also [7]). Indeed (1) is equivalent to the integral equation

$$v(\xi;x) = 1 + \int_\xi^x b(\eta)v(\eta;x)\,d\eta. \tag{11}$$

Now, the solution of (11) can be represented [4, p. 124] by

$$v(\xi;x) = 1 + \int_\xi^x b(\eta)h^*(\eta;x)\,d\eta \tag{12}$$

where

$$h^*(\xi;x) = \sum_{i=1}^{\infty} h_i(\xi;x) \tag{13}$$

with

$$h_1(\xi;x) = 1, \qquad h_{i+1}(\xi;x) = \int_\xi^x b(\eta)h_i(\eta;x)\,d\eta. \tag{14}$$

From (11) and (12) it follows that $v(\xi;x)=h^*(\xi;x)$. Thus (3) can also be written as

$$\phi(x) \leqq a(x) + \int_{x^0}^x a(\xi)b(\xi)h^*(\xi;x)\,d\xi$$

with h^* defined by (13) and (14). This agrees with the result given in [4, p. 125].

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