ON NONEQUIVALENT NORMALIZED UNCONDITIONAL BASES FOR BANACH SPACES

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Abstract. It is proved that if an infinite dimensional Banach space $X$ has an unconditional basis and is not isomorphic to $c_0$, $l_1$, or $l_2$, then $X$ has uncountably many nonequivalent normalized unconditional bases.

1. Introduction.

Definitions. Two bases $\{x_n\}$ and $\{y_n\}$ of the Banach space $X$ are called equivalent if $\sum_{n=1}^{\infty} a_n x_n$ converges if and only if $\sum_{n=1}^{\infty} a_n y_n$ converges. A basis is called conditional (unconditional) if there exists (does not exist) a series $\sum_{n=1}^{\infty} a_n x_n$ which is convergent but not unconditionally convergent. A basis $\{x_n\}$ is called normalized if $\|x_n\| = 1$ for each $n$.

This paper closes the question of how many nonequivalent, normalized bases an infinite dimensional, separable Banach space can have.

It was shown in [3, p. 18] that if an infinite dimensional Banach space has a basis then it has uncountably many nonequivalent, normalized, conditional bases. It has been proved recently that there exist separable Banach spaces which do not have bases. Therefore, the only possibilities for conditional bases are 0 or uncountably many.

It has been shown that certain infinite dimensional Banach spaces have a unique (up to equivalence) unconditional basis. See [1]. In [2] it was shown that $c_0$, $l_1$, and $l_2$ are the only such spaces.

In this paper we will prove that if a Banach space has 2 nonequivalent, normalized, unconditional bases, then it has uncountably many. Thus for unconditional bases the only possibilities are 0, 1, or uncountably many. §3 contains some remarks and an open question concerning nonpermutatively equivalent bases.

2. The main result.

Definitions. A basis $\{x_n\}$ is called symmetric if it is equivalent to each of its permutations. A sequence $\{z_n\}$ in a Banach space $X$ is called basic if it is a basis for the subspace that it spans. It is well known that each subset (in any order) of an unconditional basis is a basic sequence.

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Lemma 2.1. If \( \{x_n\} \) is an unconditional nonsymmetric basis then there exist 2 disjoint subsets of \( \{x_1, x_2, \cdots\} \) each of which is a nonsymmetric basis of its closed span.

Proof. Let \( \pi \) be the set of all permutations of the positive integers.

Let \( x = \sum_{i=1}^{\infty} f_i(x)x_i \). For each \( \sigma \in \pi \) define the permutation operator \( A_\sigma \) by \( A_\sigma(x) = \sum_{i=1}^{\infty} f_i(x)x_{\sigma(i)} \). It follows from [4, Theorem 22.1] that \( \{x_n\} \) is symmetric if and only if \( \sup_{\sigma \in \pi} \|A_\sigma\| < \infty \).

Observe that for any positive integer \( n \), there exist a finite subset \( B_n \) of the basis and a permutation \( \sigma_n \) on the subscripts of the elements in \( B_n \) such that \( \|A_{\sigma_n}\| > n \), where \( A_{\sigma_n} \) is the obvious permutation operator on the span of \( B_n \). Indeed, since the basis is not symmetric, there exist \( \sum_{i=1}^{\infty} a_i x_i \) of norm less than 1 and a permutation \( \sigma : \| \sum_{i=1}^{\infty} a_i x_{\sigma(i)} \| > n \). Pick \( \| x = \sum_{i=1}^{N} a_i x_i \| < 1 \) and \( \| \sum_{i=1}^{N} a_i x_{\sigma(i)} \| > n \). Then take \( B_n = \{x_1, x_2, \cdots, x_N\} \cup \{x_{\sigma(1)}, \cdots, x_{\sigma(n)}\} \) and take \( \sigma_n \) to be any permutation on the subscripts of \( B_n \) such that \( \sigma_n(i) = \sigma(i) \) for \( 1 \leq i \leq N \). Then \( A_{\sigma_n}(\sum_{i=1}^{N} a_i x_i) = \sum_{i=1}^{N} a_i x_{\sigma(i)} \) and thus \( A_{\sigma_n} \) has norm > \( n \).

Next we construct \( S_1, S_2, \cdots \), a disjoint sequence of finite subsets of the basis \( \{x_n\} \), such that for each \( S_n \) there exists a permutation \( \sigma_n \) on the subscripts of the elements of \( S_n \) with \( \|A_{\sigma_n}\| > n \). To do this first construct \( S_1 \) by the previous method. Assume \( S_1, \cdots, S_{n-1} \) have been constructed. Then \( \{x_i\}_{i=1}^{\infty} \sim S_1 \sim \cdots \sim S_{n-1} \) is still a nonsymmetric basis for we have discarded only a finite number of elements. Apply the previous method.

Now we can complete the proof of the lemma. \( \bigcup_{n=1}^{\infty} S_{2n-1} \) and \( \bigcup_{n=1}^{\infty} S_{2n} \) are both nonsymmetric, since there is no uniform bound on the permutation operators for either basis.

This lemma can be sharpened by applying it successively.

Proposition 2.1. If \( \{x_i\} \) is an unconditional, nonsymmetric basis, then there exists \( B_1, B_2, \cdots \) a disjoint sequence of nonsymmetric subsets of the basis.

Theorem 2.1. Let an infinite dimensional Banach space \( X \) have an unconditional basis and not be isomorphic to \( c_0 \), \( l_1 \), or \( l_2 \). Then \( X \) has an uncountable number of nonequivalent, normalized unconditional bases.

Proof. Case 1. \( \{x_n\} \) is not symmetric. We will construct uncountably many nonequivalent permutations of \( \{x_n\} \).

Let \( B_1, B_2, \cdots \) be disjoint nonsymmetric subsets of the basis. Without loss of generality we can assume \( \bigcup_{n=1}^{\infty} B_n = \{x_1, x_2, \cdots\} \). For each \( B_n \) select 2 enumerations of \( B_n \) which give rise to nonequivalent bases. Call them the 0th and 1st enumerations.
Let $\bigcup_{k=1}^{\infty} W_k = \{1, 2, 3, \cdots\}$ be a disjoint union with each $W_k$ infinite. Let $\{w_{k1}, w_{k2}, \cdots\}$ be the increasing enumeration of $W_k$. To each binary sequence of 0’s and 1’s we can associate a permutation $\{y_n\}$ of the original basis $\{x_n\}$ as follows: if there is a 0 in the $k$th place of the binary sequence take $y_{w_{k1}}, y_{w_{k2}}, \cdots$ to be the 0th enumeration of $B_k$; if there is a 1 then use the 1st enumeration of $B_k$. Observe that different binary sequences give rise to nonequivalent bases.

Case 2. $\{x_n\}$ is symmetric. Since $X$ is not isomorphic to $c_0$, $l_1$, or $l_2$, there exists a nonequivalent basis $\{d_n\}$. We can assume $\{d_n\}$ is symmetric, otherwise we would be done by Case 1.

Since $X \cong X \oplus X$, $X$ has a basis equivalent to $\{x_1, d_1, x_2, d_2, \cdots\}$. Let $\bigcup_{k=1}^{\infty} W_k = \{1, 2, 3, \cdots\}$ be a disjoint union with each $W_k$ infinite. To each binary sequence which does not terminate in all 0’s or 1’s we can associate a permutation of $\{x_1, d_1, x_2, d_2, \cdots\}$, such that distinct binary sequences give rise to nonequivalent bases.

3. Some remarks. (a) Two bases $\{x_n\}$ and $\{y_n\}$ of a Banach space $X$ are called permutatively equivalent if there exists a permutation $\sigma$ of the positive integers: $\{x_{\sigma(n)}\}$ and $\{y_n\}$ are equivalent.

(b) $c_0$, $l_1$, and $l_2$ are not the only spaces which have a unique normalized unconditional basis (up to permutative equivalence), $l_1 \times l_2$ also has this property. See [4, p. 547].

(c) Each $l^p$, $p \neq 1$, 2, and $L^p[0, 1]$, $p \neq 2$, has two normalized unconditional bases which are not permutatively equivalent. See [4, Theorem 18.3 and 18.4]. Note that for $l^p$, $p \neq 1$, 2, the canonical basis and a non-equivalent basis would automatically be permutatively nonequivalent.

(d) The question of how many permutatively nonequivalent, normalized unconditional bases a Banach space can have is open.

REFERENCES


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