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ON THE BOUNDEDNESS OF PSEUDO-DIFFERENTIAL OPERATORS

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ABSTRACT. In this note, we use the sequence version of Cotlar's lemma and a partition of unity to give a proof of the L^2-boundedness of a class of pseudo-differential operators.

Introduction and results. Let $p(x, \xi)$ be a continuous function on $R^n \times R^n$. Then, a pseudo-differential operator P with the symbol $p(x, \xi)$ is a linear map of $C_0^\infty(R^n)$ into $C^0(R^n)$, defined by

$$P_u = ((2\pi)^{1/2})^{-n} \int e^{ix\xi} p(x, \xi) \hat{u}(\xi)\, d\xi$$

for $u \in C_0^\infty(R^n)$, where $\hat{u}$ is the Fourier transform of u.

In [1] A. Calderón and R. Vaillencourt prove the following

THEOREM A. *Let $p(x, \xi)$ be a function defined on $R^n \times R^n$ such that*

$$|(1 + \partial_{x_n})^3 \cdots (1 + \partial_{x_1})^3 (1 + \partial_{\xi_n})^3 \cdots (1 + \partial_{\xi_1})^3 p(x, \xi)| \leqq C$$

for all $(x, \xi) \in R^n \times R^n$. Then the pseudo-differential operator associated with the symbol $p(x, \xi)$ can be extended to a bounded operator from $L^2(R^n)$ to $L^2(R^n)$.

Using the sequence version of Cotlar's lemma (cf. [2]) and a partition of unity, we can prove the following result analogous to Theorem A.

THEOREM. *Let $p(x, \xi)$ be a function defined on $R^n \times R^n$ such that*

$$(1) \qquad \int_Q |\partial_x^\alpha \partial_\xi^\beta p(x, \xi)|\, dx \leqq C, \quad \textit{for } 0 \leqq \alpha_i \leqq 2, 0 \leqq \beta_j \leqq 3,$$

and all $(x, \xi) \in Q \times R^n$, where Q is any cube with edges of length two and parallel to the axes. Then the associated operator can be extended to a bounded operator from $L^2(R^n)$ to $L^2(R^n)$.

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REMARK. Actually a slight modification of the proof shows that the theorem still holds under the following weaker assumptions on $p(x, \xi)$ instead of (1):

$$\int_Q |\partial_x^\alpha \partial_\xi^\beta p(x, \xi)| \leqq C$$

and

$$\int_Q |\partial_x^\alpha \partial_\xi^\beta p(x + h, \xi) - \partial_x^\alpha \partial_\xi^\beta p(x, \xi)| \, dx \leqq Ch^\delta$$

for some $\delta>0$, $0\leqq\alpha_i\leqq 1$, $0\leqq\beta_j\leqq 3$, $0\leqq h\leqq 1$ and all $(x, \xi)\in Q\times R^n$.

PROOF OF THEOREM. It suffices to prove the theorem for $n=1$. We use C to denote various constants. Also, we can assume without loss of generality that $p(x, \xi)$ has compact support in ξ. Let $f(x)$ be an infinitely differentiable function with $\operatorname{supp} f\subset C(-5/4, 5/4)$ and equal to one for $|x|\leqq 1$. We define

$$p_i(x, \xi) = \left(f(x - i)\Big/\sum_{-\infty}^{\infty} f(x - k)\right) p(x, \xi)$$

for all integers i. It is easy to show that

$$\int_R |\partial_x^\alpha \partial_\xi^\beta p_i(x, \xi)| \, dx \leqq C$$

for $0\leqq\alpha\leqq 2$, $0\leqq\beta\leqq 3$ and all $(x, \xi)\in R\times R$.

As the Fourier transform of differentiation is multiplication, we can conclude that

$$|\partial_\xi^\beta \tilde{p}_i(\eta, \xi)| \leqq C/1 + |\eta|^2 \quad \text{for } 0 \leqq \beta \leqq 3 \tag{5}$$

for all $(\eta, \xi)\in R\times R$, where $\tilde{p}_i(\eta, \xi)$ is the Fourier transform of $p_i(x, \xi)$ in the space variable x.

As in Kohn-Nirenberg [3], we then have

$$\|P_i^{(\beta)} u\| \leqq C\|u\| \quad \text{for } 0 \leqq \beta \leqq 3,$$

where $P_i^{(\beta)}$ is the operator associated with the symbol $\partial_\xi^{(\beta)} p_i(x, \xi) = p_i^{(\beta)}(x, \xi)$.

Since $p_{2n+1}(x, \xi)$ and $p_{2m+1}(x, \xi)$ have disjoint support in x for $n\neq m$, we have

$$P_{2n+1}^* P_{2m+1} = 0 \tag{6}$$

where P^* is the adjoint of P; that is

$$(Pu, v) = (u, P^* v) \quad \text{for } u, v \in C_0^\infty(R^1),$$

or

$$P^*v(\xi) = \int e^{-ix\xi}\overline{p(x,\xi)}v(x)\,dx. \tag{7}$$

Let u and v be $C_0^\infty(R)$ functions and $I=(P_{2n+1}^*u, P_{2m+1}^*v)$. Applying Parseval's theorem and using (7) we obtain

$$\begin{aligned} I &= \int u(x)\overline{v(y)}\,dx\,dy \int e^{i(y-x)\xi}\overline{p_{2n+1}(x,\xi)}p_{2m+1}(y,\xi)\,d\xi \\ &= -\int \frac{u(x)\overline{v(y)}}{(x-y)^3}\,dx\,dy \int e^{i(y-x)\xi}\frac{\partial^3}{\partial\xi^3}\overline{p_{2n+1}(x,\xi)}p_{2m+1}(y,\xi)\,d\xi \\ &= \sum_{\alpha+\beta=3} C_{\alpha,\beta}\int d\xi \int \hat{u}(n)\,dn \int \overline{\hat{v}(\zeta)}\,d\zeta \int H(x,y,\xi,\eta,\zeta)\,dx\,dy \end{aligned} \tag{8}$$

with

$$\begin{aligned} H = {} & \frac{e^{i(\xi-\zeta)y}}{1+(\xi-\eta)^2}\frac{e^{i(\eta-\xi)x}}{1+(\zeta-\xi)^2}\left(1-\frac{\partial^2}{\partial x^2}\right) \\ & \cdot\left(1-\frac{\partial^2}{\partial y^2}\right)\frac{p_{2m+1}^{(\beta)}(y,\xi)p_{2n+1}^{(\alpha)}(x,\xi)}{(x-y)^3}. \end{aligned}$$

As $1/(x-y)^k \leqq C/(m-n)^3+1$ for $k\geqq 3$, $x\in \operatorname{supp} p_{2n+1}^{(\alpha)}(x,\xi)$ and $y\in \operatorname{supp} p_{2m+1}^{(\beta)}(y,\xi)$ with $n\neq m$, we can conclude from (8) that

$$\begin{aligned} |I| &\leqq \left[\int d\xi \int \frac{\hat{u}(n)}{1+(\xi-n)^2}\,dn \int \frac{|\hat{v}(\zeta)|}{1+(\xi-\zeta)^2}\,d\zeta\right]\frac{c}{(m-n)^3+1} \\ &\leqq \frac{c}{(m-n)^3+1}\,\|u\|\,\|v\|. \end{aligned} \tag{9}$$

In virtue of (6) and (9), we can apply Cotlar's lemma to deduce the fact that

$$\left\|\sum_{n=1}^{\infty} P_{2n+1}\right\| \leqq C. \tag{10}$$

Similarly, we can prove

$$\left\|\sum_{n=0}^{\infty} P_{2n}\right\| \leqq C. \tag{11}$$

(10) and (11) imply that P can be extended to a bounded operator from L^2 to L^2.

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