CLOSING STABLE AND UNSTABLE MANIFOLDS
ON THE TWO SPHERE

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Abstract. Let f be a diffeomorphism of the two sphere. In this note we prove that if the unstable manifold of a fixed point p for f accumulates on the stable manifold of p, then f can be approximated arbitrarily closely C¹, r≥1, such that they intersect.

1. The results. Let d be a distance on the two sphere S² coming from a Riemannian metric. The stable manifold of a point p is defined to be

Wˢ(p,f) = {x ∈ S² : d(fⁿx, fⁿp) → 0 as n → ∞}.

The unstable manifold of p is the stable manifold for f⁻¹, Wᵘ(p,f) = Wˢ(p,f⁻¹). Let Wˢ(p,f)−{p}=Wˢ(p,f)' and Wᵘ(p,f)−{p}=Wᵘ(p,f)'. A fixed point p is called a saddle point if the eigenvalues λ, µ of the derivative Df(p) satisfy 0<|λ|<1<|µ|.

Theorem. Let f be a C¹ diffeomorphism of S², r=1, and p a fixed saddle point of f such that Wˢ(p,f)' ∩ closure Wᵘ(p,f)≠∅. Then f can be approximated arbitrarily closely C¹ by f' such that f'(p)=p and Wˢ(p,f')' ∩ Wᵘ(p,f')≠∅.

Corollary. There is a residual subset (complement of a first category set) R of the set of all C¹ diffeomorphisms, Diff¹(S²), such that if f ∈ R, p is a saddle fixed point of f, and Wˢ(p,f)' ∩ closure Wᵘ(p,f)≠∅ then Wˢ(p,f)' ∩ Wᵘ(p,f)≠∅.

The reason for restricting to S² is to use the Jordan separation theorem. Therefore analogous results are true on the two disk and R². In the case of the disk, the above theorem should prove useful to prove a conjecture of Smale in [6]. For this use it would be nice to prove the result assuming only that {p}≠ closure Wˢ(p,f) ∩ closure Wᵘ(p,f).

Next, I would hope the results would be true for periodic points but was unable to prove it. There are all sorts of related closing lemma conjectures. See [5].
The only real examples I know of that satisfy the hypothesis of the theorems are cycles. The theorem was proved for these by Palls in [3]. I used this type of thinking to develop the present proof.

I would like to thank S. Newhouse who introduced me to this problem many years ago.

2. Proof of theorem. Assume \( W^s(p, f)' \cap \text{closure} \ W^u(p, f) \neq \emptyset \) but \( W^s(p, f)' \cap \text{closure} \ W^u(p, f) = \emptyset \). (Otherwise we are done.) Let

\[
W^s_\varepsilon(p) = \{ x \in W^s(p, f) : d(f^n x, p) \leq \varepsilon \text{ for } n \geq 0 \}.
\]

This is the local stable manifold. Similarly,

\[
W^u_\varepsilon(p) = \{ x \in W^u(p, f) : d(f^n x, p) \leq \varepsilon \text{ for } n \leq 0 \}.
\]

\( W^s(p, f)' \), closure \( W^u(p, f) \), and \( W^a(p, f)' \cap \text{closure} \ W^u(p, f) \) are invariant under \( f \) so \( W^s_\varepsilon(p, f)' \cap \text{closure} \ W^u(p, f) \neq \emptyset \) for every \( \varepsilon > 0 \).

Let the eigenvalues of \( Df(p) \) be \( \mu, \lambda \) where \( 0 < |\lambda| < 1 < |\mu| \). Assume \( Df(p) \) preserves the orientation of both the stable and unstable direction so \( 0 < \lambda < 1 < \mu \). We make the modifications for the other cases below. Take \( y \in W^s_\varepsilon(p, f)' \cap \text{closure} \ W^u(p, f) \). We want to construct a fundamental neighborhood of \( y \). Let \( T \) be a transversal to \( W^s_\varepsilon(p, f) \) such that \( y \) lies between \( T \cap W^s_\varepsilon(p, f) \) and \( f(T) \cap W^s_\varepsilon(p, f) \). Connect the two ends of \( T \) to the two ends of \( f(T) \) to enclose a region \( Q \) such that \( y \in \text{interior} \ Q, Q \cap f(Q) = f(T) \), and \( f^j(y) \notin Q \) for \( j \geq 1 \).

\[
\begin{array}{c}
T \\
\bullet y \\
\text{Q} \\
fT
\end{array}
\]

Such a region can easily be constructed in local coordinates that linearize \( f \) near \( p \) given by Hartman’s theorem.

Let \( x_n \) be the point of \( f^n W^u_\varepsilon(p, f) \) that is closest to \( y \). Since

\[
\bigcup \{ f^n W^u_\varepsilon(p, f) : n \geq 0 \} = W^u(p, f)
\]

we have that \( x_n \) converges to \( y \). Let \( L_0 = W^s_\delta(p, f) \) be such that \( L_0 \cap T \)

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is one point. Let $L=fL_0$. Let $N$ be an integer such that, for $n \geq N$, $d(y, x_n) < d(y, \partial Q)$ and $d(y, x_n) < d(y, L)$. Let $\mathcal{O}_-(z) = \{f^{-i}(z) : 0 \leq i\}$ be the backward orbit of $z$. For $n \geq N$, $x_n \in \mathcal{O}_-(x_n) \cap \text{interior } Q$.

**Lemma.** For $n \geq N$, $x_n = \mathcal{O}_-(x_n) \cap \text{interior } Q$.

**Proof.** For $z \in W^u(p, f)$ let $[p; z]$ be the part of $W^u(p, f)$ between $p$ and $z$ including both end points. Assume $q = f^{-j}(x_n) \in \text{interior } Q$ for $j \geq 1$. Let $Q'$ be the component of $Q - L_0$ such that $q \in Q'$. Let $r$ be the first point as we move from $p$ to $q$ along $[p, q]$ such that $r \in Q'$. We want to show $f^i[p, r]$ contains a point closer to $y$ than $f^i(q) = x_n$. $f^i[p, q] \subset f^i[p, q] = [p, x_n] \subset f^nW^u(p, f)$ so this contradicts definition of $x_n$. Therefore it contradicts $f^{-i}(q) \in \text{interior } Q$ and proves the lemma.

Let $S_0$ be the part of the boundary of $Q'$ such that $P = S_0 \cup L_0 \cup [p, r]$ is a simple closed curve. Let $D_0$ be the disk bounded by $P$ such that $Q' \subset D_0$. Here we use Jordan separation theorem on the two sphere. $fQ' \subset D_0$ because $\partial D_0$ does not cut across $fQ'$ (by choice of $r$). Let $D_1 = fD_0$ and $D_k = \cup \{f^i(D_1) : 0 \leq i \leq k-1\}$. $D_1 \neq D_1$. Let $S = fS_0$. $\partial D_1 = S \cup L \cup [p, fr]$. The reader can check that for $k \geq 2$

$$\partial D_k - S - L \subset f(\partial D_{k-1} - S - L)$$

and hence by induction

$$\partial D_j - S - L \subset f^{j-1}(\partial D_1 - S - L) \subset f^{j-1}[p, fr] = f^j[p, r]$$

or

$$\partial D_j \subset S \cup L \cup f^j[p, r]$$

This boundary never crosses $L_0 - L$ so $L_0 - L$ all lies on the same side of $\partial D_j$. This is the opposite side from $f(Q') \subset D_j$. Therefore $y \in L_0 - L$ is not in $D_j$. The point of $D_j$ closest to $y$ must lie on $\partial D_j$. $x_n = f^j(q) \in \text{interior } D_j$ is closer to $y$ than $S$ and $L$ so the closest point is on $f^j[p, r]$. This is closer to $y$ than $x_n$. This completes the proof of the lemma.

**Added in Proof.** It is not necessary that $D_j$ be a disk for $j \geq 2$. However, $L_0 - L$ lies in one component of $S^2 - \partial D_j$. This is a different component from $f(Q') \subset D_j$.

Thus we have constructed a sequence $x_n \in \text{interior } Q$, $n \geq N$, that converges to $y$, and $\mathcal{O}_-(x_n) \cap \text{interior } Q = x_n$. Also $\mathcal{O}_+(y) \cap \text{interior } Q = y$.

Now assume we are given $\varepsilon > 0$ and want to approximate $f C^r$ within $\varepsilon$. By a standard argument, see [1], there exists $\delta > 0$ such that if $z \in Q$ and $d(y, z) < \delta$, then there exists a function $f'$ within $\varepsilon$ of $f$ in the $C^r$ topology such that $f'(z) = f(y)$ and $f(x) = f'(x)$ for $x \notin Q$. Take $n \geq N$ large enough
such that $d(y, x_n) < \delta$. Take $f'$ as above. $p \notin Q$ so $f'(p) = f(p) = p$ is a fixed point. $f''(x_n) = f(y)$. By induction, $(f')^k(x_n) = f^k(y) \notin Q$. Therefore $(f')^k(x_n)$ converges to $p$ as $j$ goes to infinity. Therefore $x_n \in W^u(p, f')$. $f^{-1}(x) = (f')^{-1}(x)$ for $x \notin f(Q)$. By induction $(f')^{-k}(x_n) = f^{-k}(x_n) \notin f(Q)$. Therefore $(f')^{-k}(x_n)$ converges to $p$ and $x_n \in W^u(p, f')$. This completes the proof in the case that $f$ preserves orientation.

If either eigenvalue of $f$ is negative we need to modify the proof. Let $g = f^2$. Take a transversal $T$ and use $T$ and $g(T)$ to construct the neighborhood $Q$ of $y$. $Q_0 = Q - f(Q)$. Then $f^j(y) \notin Q_0$ for $j \geq 1$. Let $L_0$ be as before and $L = gL_0$. Take $N$ such that $d(x_n, Q) < d(y, \partial Q)$, $d(y, \partial Q_0)$, $d(y, L)$.

In the lemma assume $q = f^{-1}(x_n) \in \partial Q_0$. $x_n \in Q_0$ and local analysis about $p$ shows $j \geq 1$. Let $q, r, S_0, D_0$ be as before.

$$D_2 = gD_0, \quad D_{2k} = \bigcup \{g^i(D_2): 0 \leq i \leq k - 1\},$$

$$D_{2k+1} = fD_{2k}, \quad \text{and} \quad S = gS_0.$$ 

$$\partial D_2 = S \cup L \cup [p, fr].$$ 

$$\partial D_{2k} = S \cup L \cup f^{2k}[p, r],$$ 

$$\partial D_{2k+1} = f(S) \cup f(L) \cup f^{2k+1}[p, r].$$

$x_n \in \text{interior } D_j$ so we get a contradiction as before to prove $\partial(x_n) \cap \text{interior } Q_0 = x_n$.

When we construct $f'$ we require $f(x) = f'(x)$ for $x \notin Q_0$ and $f'(x_n) = f(y)$. $\partial_+(y) \cap Q_0 = y$ and $\partial_-(x_n) \cap Q_0 = x_n$ so as before $x_n \in W^u(p, f') \cap W^u(p, f')$. 

3. **Proof of corollary.** We construct a semicontinuous function and use the method of Pugh [4] to prove the corollary.

Let $P(f) = \{p \in S^2: p$ is a saddle fixed point of $f$ and $W^s(p, f')$ and $W^u(p, f)$ intersect transversally}. Let $F$ be the family of closed subsets of $S^2$ with the Hausdorff metric. $P : \text{Diff}^r(S^2) \to F$. Let $\text{Diff}^r(S^2)$ have the usual uniform $C^r$ topology. By the persistence of transverse intersection [1], $P$ is lower semicontinuous. The points at which $P$ is continuous forms a residual subset $R \subset \text{Diff}^r(S^2)$, see [2].

We claim $R$ satisfies the corollary. Assume not and there exist $f \in R$, and $p$ a saddle fixed point of $f$ such that $W^s(p, f') \cap \text{closure } W^u(p, f) \neq \emptyset$ and $W^s(p, f') \cap W^u(p, f) = \emptyset$. Let $N$ be any $C^r$ neighborhood of $f$. By the theorem we can approximate $f$ by $f' \in N$ such that $p$ is a saddle fixed point of $f'$ and $W^s(p, f') \cap W^u(p, f') \neq \emptyset$. Then we can approximate $f'$ by $f'' \in N$ such that this intersection is transverse. Therefore $p \in P(f'')$. Since $N$ is an arbitrary neighborhood of $f$ and $P$ is continuous at $f$ we get $p \in P(f)$. Contradiction. 

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