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CLOSING STABLE AND UNSTABLE MANIFOLDS ON THE TWO SPHERE

CLARK ROBINSON[1]

ABSTRACT. Let f be a diffeomorphism of the two sphere. In this note we prove that if the unstable manifold of a fixed point p for f accumulates on the stable manifold of p, then f can be approximated arbitrarily closely C^r, $r\geqq 1$, such that they intersect.

1. **The results.** Let d be a distance on the two sphere S^2 coming from a Riemannian metric. The *stable manifold* of a point p is defined to be

$$W^s(p,f) = \{x \in S^2 : d(f^n x, f^n p) \to 0 \text{ as } n \to \infty\}.$$

The *unstable manifold* of p is the stable manifold for f^{-1}, $W^u(p,f)= W^s(p,f^{-1})$. Let $W^s(p,f)-\{p\}=W^s(p,f)'$ and $W^u(p,f)-\{p\}=W^u(p,f)'$. A fixed point p is called a *saddle point* if the eigenvalues λ, μ of the derivative $Df(p)$ satisfy $0<|\lambda|<1<|\mu|$.

THEOREM. *Let f be a C^r diffeomorphism of S^2, $r\geqq 1$, and p a fixed saddle point of f such that $W^s(p,f)' \cap \text{closure } W^u(p,f)\neq\varnothing$. Then f can be approximated arbitrarily closely C^r by f' such that $f'(p)=p$ and $W^s(p,f')' \cap W^u(p,f')\neq\varnothing$.*

COROLLARY. *There is a residual subset (complement of a first category set) R of the set of all C^r diffeomorphisms,* $\text{Diff}^r(S^2)$, *such that if $f\in R$, p is a saddle fixed point of f, and $W^s(p,f)' \cap \text{closure } W^u(p,f)\neq\varnothing$ then $W^s(p,f)' \cap W^u(p,f)\neq\varnothing$.*

The reason for restricting to S^2 is to use the Jordan separation theorem. Therefore analogous results are true on the two disk and R^2. In the case of the disk, the above theorem should prove useful to prove a conjecture of Smale in **[6]**. For this use it would be nice to prove the result assuming only that $\{p\}\neq \text{closure } W^s(p,f) \cap \text{closure } W^u(p,f)$.

Next, I would hope the results would be true for periodic points but was unable to prove it. There are all sorts of related closing lemma conjectures. See **[5]**.

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The only real examples I know of that satisfy the hypothesis of the theorems are cycles. The theorem was proved for these by Palls in [3]. I used this type of thinking to develop the present proof.

I would like to thank S. Newhouse who introduced me to this problem many years ago.

2. **Proof of theorem.** Assume $W^s(p,f)' \cap \text{closure } W^u(p,f) \neq \varnothing$ but $W^s(p,f)' \cap W^u(p,f) = \varnothing$. (Otherwise we are done.) Let

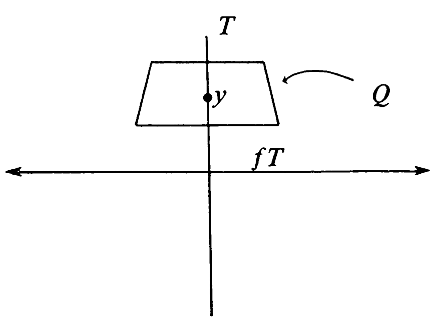
$$W^s_\varepsilon(p) = \{x \in W^s(p,f) : d(f^n x, p) \leqq \varepsilon \text{ for } n \geqq 0\}.$$

This is the local stable manifold. Similarly,

$$W^u_\varepsilon(p) = \{x \in W^u(p,f) : d(f^n x, p) \leqq \varepsilon \text{ for } n \leqq 0\}.$$

$W^s(p,f)'$, closure $W^u(p,f)$, and $W^s(p,f)' \cap \text{closure } W^u(p,f)$ are invariant under f so $W^s_\varepsilon(p,f)' \cap \text{closure } W^u(p,f) \neq \varnothing$ for every $\varepsilon > 0$.

Let the eigenvalues of $Df(p)$ be μ, λ where $0 < |\lambda| < 1 < |\mu|$. Assume $Df(p)$ preserves the orientation of both the stable and unstable direction so $0 < \lambda < 1 < \mu$. We make the modifications for the other cases below. Take $y \in W^s_\varepsilon(p,f)' \cap \text{closure } W^u(p,f)$. We want to construct a fundamental neighborhood of y. Let T be a transversal to $W^s_\varepsilon(p,f)$ such that y lies between $T \cap W^s_\varepsilon(p,f)$ and $f(T) \cap W^s_\varepsilon(p,f)$. Connect the two ends of T to the two ends of $f(T)$ to enclose a region Q such that $y \in \text{interior } Q$, $Q \cap f(Q) = f(T)$, and $f^j(y) \notin Q$ for $j \geqq 1$.



Such a region can easily be constructed in local coordinates that linearize f near p given by Hartman's theorem.

Let x_n be the point of $f^n W^u_1(p,f)$ that is closest to y. Since

$$\bigcup \{f^n W^u_1(p,f) : n \geqq 0\} = W^u(p,f)$$

we have that x_n converges to y. Let $L_0 = W^s_\delta(p,f)$ be such that $L_0 \cap T$

is one point. Let $L=fL_0$. Let N be an integer such that, for $n\geqq N$, $d(y, x_n)<d(y, \partial Q)$ and $d(y, x_n)<d(y, L)$. Let $\mathcal{O}_-(z)=\{f^{-j}(z):0\leqq j\}$ be the backward orbit of z. For $n\geqq N$, $x_n \in \mathcal{O}_-(x_n) \cap$ interior Q.

LEMMA. *For* $n\geqq N$, $x_n=\mathcal{O}_-(x_n) \cap$ *interior* Q.

PROOF. For $z \in W^u(p,f)$ let $[p; z]$ be the part of $W^u(p,f)$ between p and z including both end points. Assume $q=f^{-j}(x_n) \in$ interior Q for $j\geqq 1$. Let Q' be the component of $Q-L_0$ such that $q \in Q'$. Let r be the first point as we move from p to q along $[p, q]$ such that $r \in Q'$. We want to show $f^j[p, r]$ contains a point closer to y than $f^j(q)=x_n$. $f^j[p, r] \subset f^j[p, q]=[p, x_n] \subset f^n W_1^u(p,f)$ so this contradicts definition of x_n. Therefore it contradicts $f^{-j}(q) \in$ interior Q and proves the lemma.

Let S_0 be the part of the boundary of Q' such that $P=S_0 \cup L_0 \cup [p, r]$ is a simple closed curve. Let D_0 be the disk bounded by P such that $Q' \subset D_0$. Here we use Jordan separation theorem on the two sphere. $fQ' \subset D_0$ because ∂D_0 does not cut across fQ' (by choice of r). Let $D_1=fD_0$ and $D_k=\bigcup \{f^i(D_1):0\leqq i\leqq k-1\}$. $y \notin D_1$. Let $S=fS_0$. $\partial D_1=S\cup L\cup [p, fr]$. The reader can check that for $k\geqq 2$

$$\partial D_k - S - L \subset f(\partial D_{k-1} - S - L)$$

and hence by induction

$$\partial D_j - S - L \subset f^{j-1}(\partial D_1 - S - L) \subset f^{j-1}[p, fr] = f^j[p, r]$$

or

$$\partial D_j \subset S \cup L \cup f^j[p, r].$$

This boundary never crosses L_0-L so L_0-L all lies on the same side of ∂D_j. This is the opposite side from $f(Q') \subset D_j$. Therefore $y \in L_0-L$ is not in D_j. The point of D_j closest to y must lie on ∂D_j. $x_n=f^j(q) \in$ interior D_j is closer to y than S and L so the closest point is on $f^j[p, r]$. This is closer to y than x_n. This completes the proof of the lemma. □

ADDED IN PROOF. It is not necessary that D_j be a disk for $j\geqq 2$. However, L_0-L lies in one component of $S^2-\partial D_j$. This is a different component than $f(Q') \subset D_j$.

Thus we have constructed a sequence $x_n \in$ interior Q, $n\geqq N$, that converges to y, and $\mathcal{O}_-(x_n) \cap$ interior $Q=x_n$. Also $\mathcal{O}_+(y) \cap$ interior $Q=y$.

Now assume we are given $\varepsilon>0$ and want to approximate f C^r within ε. By a standard argument, see [**1**], there exists $\delta>0$ such that if $z \in Q$ and $d(y, z)<\delta$, then there exists a function f' within ε of f in the C^r topology such that $f'(z)=f(y)$ and $f(x)=f'(x)$ for $x \notin Q$. Take $n\geqq N$ large enough such that $d(y, x_n)<\delta$. Take f' as above. $p \notin Q$ so $f'(p)=f(p)=p$ is a fixed point. $f'(x_n)=f(y)$. By induction, $(f')^k(x_n)=f^k(y) \notin Q$. Therefore $(f')^k(x_n)$ converges to p as j goes to infinity. Therefore $x_n \in W^s(p,f')$. $f^{-1}(x)=(f')^{-1}(x)$ for $x \notin f(Q)$. By induction $(f')^{-k}(x_n)=f^{-k}(x_n) \notin f(Q)$. Therefore $(f')^{-k}(x_n)$ converges to p and $x_n \in W^u(p,f')$. This completes the proof in the case that f preserves orientation.

If either eigenvalue of f is negative we need to modify the proof. Let $g=f^2$. Take a transversal T and use T and $g(T)$ to construct the neighborhood Q of y. $Q_0=Q-f(Q)$. Then $f^j(y) \notin Q_0$ for $j \geqq 1$. Let L_0 be as before and $L=gL_0$. Take N such that $d(x_n, y)<d(y, \partial Q), d(y, \partial Q_0), d(y, L)$.

In the lemma assume $q=f^{-j}(x_n) \in$ interior Q_0. $x_n \in Q_0$ and local analysis about p shows $j \geqq 1$. Let q, r, S_0, D_0 be as before. Let

$$D_2 = gD_0, \qquad D_{2k} = \bigcup \{g^i(D_2):0 \leqq i \leqq k-1\},$$

$$D_{2k+1} = fD_{2k}, \quad \text{and} \quad S = gS_0.$$

$$\partial D_2 = S \cup L \cup [p, fr].$$

$$\partial D_{2k} \subset S \cup L \cup f^{2k}[p, r],$$

$$\partial D_{2k+1} \subset f(S) \cup f(L) \cup f^{2k+1}[p, r].$$

$x_n \in$ interior D_j so we get a contradiction as before to prove $\mathcal{O}_-(x_n) \cap$ interior $Q_0=x_n$.

When we construct f' we require $f(x)=f'(x)$ for $x \notin Q_0$ and $f'(x_n)=f(y)$. $\mathcal{O}_+(y) \cap Q_0=y$ and $\mathcal{O}_-(x_n) \cap Q_0=x_n$ so as before $x_n \in W^s(p,f') \cap W^u(p,f')$. $\square$

3. **Proof of corollary.** We construct a semicontinuous function and use the method of Pugh [4] to prove the corollary.

Let $P(f)=\{p \in S^2 : p$ is a saddle fixed point of f and $W^s(p,f)'$ and $W^u(p,f)$ intersect transversally$\}$. Let F be the family of closed subsets of S^2 with the Hausdorff metric. $P:\mathrm{Diff}^r(S^2) \to F$. Let $\mathrm{Diff}^r(S^2)$ have the usual uniform C^r topology. By the persistence of transverse intersection [1], P is lower semicontinuous. The points at which P is continuous forms a residual subset $R \subset \mathrm{Diff}^r(S^2)$, see [2].

We claim R satisfies the corollary. Assume not and there exist $f \in R$, and p a saddle fixed point of f such that $W^s(p,f)' \cap$ closure $W^u(p,f) \neq \varnothing$ and $W^s(p,f)' \cap W^u(p,f)=\varnothing$. Let N be any C^r neighborhood of f. By the theorem we can approximate f by $f' \in N$ such that p is a saddle fixed point of f' and $W^s(p,f')' \cap W^u(p,f') \neq \varnothing$. Then we can approximate f' by $f'' \in N$ such that this intersection is transverse. Therefore $p \in P(f'')$. Since N is an arbitrary neighborhood of f and P is continuous at f we get $p \in P(f)$. Contradiction. $\square$

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Department of Mathematics, Northwestern University, Evanston, Illinois 60201