COMMON FIXED POINTS FOR SEMIGROUPS OF MAPPINGS
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Abstract. Let \( X \) be a compact convex subset of a strictly convex Banach space. Let \( S \) be a Hausdorff topological semigroup which is either left amenable or left reversible. Then for any generalised nonexpansive (jointly) continuous action of \( S \) on \( X \), \( X \) contains a common fixed point of \( S \).

1. Introduction. Let \( S \) be a (nonempty) topological semigroup, i.e. \( S \) is a semigroup with a Hausdorff topology such that for each \( s \) in \( S \), the mappings \( x \rightarrow sx \) and \( x \rightarrow xs \) of \( S \) into \( S \) are continuous. Let \( C(S) \) be the Banach algebra of all bounded real-valued continuous functions on \( S \) with the supremum norm. A function \( f \) in \( C(S) \) is left uniformly continuous if the mapping \( s \rightarrow l_s f \), where \( l_s f(t) = f(st) \) for all \( s, t \in S \), is continuous on \( S \) ([12], [13]). Let \( \text{LUC}(S) \) be the family of all left uniformly continuous functions on \( S \). Then \( \text{LUC}(S) \) is a Banach subalgebra of \( C(S) \) which contains all of the real-valued constant functions on \( S \) and is translation invariant [13]. \( S \) is left amenable if \( \text{LUC}(S) \) has a left invariant mean \( \mu \), i.e. \( \mu \) is a continuous linear functional on \( \text{LUC}(S) \) such that \( \| \mu \| = \mu(e) = 1 \) and \( \mu(l_s f) = \mu(f) \) for all \( x \in S, f \in \text{LUC}(S) \), where \( e \) is the function with \( e(s) = 1 \) for all \( s \in S \) [13]. When \( S \) is discrete, this definition coincides with that of M. M. Day in [1].

Let \( X \) be a subset of a Banach space \( B \) with norm \( p \). An action of \( S \) on \( X \) is a mapping of \( S \times X \) into \( X \), denoted by \( (s, x) \rightarrow sx \) such that \( (st)x = s(tx) \) for all \( s, t \in S, x \in X \) (as a consequence, \( S \) can be considered as a family of functions of \( X \) into \( X \) with the possibility that different elements in \( S \) correspond to the same function). A point \( x \) in \( X \) is a common fixed point of \( S \) (with respect to an action) if \( sx = x \) for all \( s \in S \). An action

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of $S$ on $X$ is nonexpansive if $p(sx - sy) \leq p(x - y)$ for all $s \in S$, $x, y \in X$; it is generalized nonexpansive if there exist nonnegative real numbers $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ such that $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 = 1$ and for all $s \in S$, $x, y \in X$,
\[
p(sx - sy) \leq \alpha_1 p(x - sx) + \alpha_2 p(y - sy) + \alpha_3 p(x - sy) + \alpha_4 p(y - sx) + \alpha_5 p(x - y).
\]
It is obvious that an action of $S$ on $X$ is generalized nonexpansive if it is nonexpansive. The converse is not true even for the case when $X$ is a bounded closed interval [17].

A Banach space $B$ with norm $p$ is strictly convex if for any $x, y, z$ in $B$, $p(x - z) + p(z - y) = p(x - y)$ implies that $z \in [x, y] = \{(1 - t)x + ty: t \in [0, 1]\}$. It is the main purpose of this paper to prove the following result. Related results for family of nonexpansive mappings can be found in [3], [7], [11], [15] and [16]. The notion of generalized nonexpansive mappings for metric spaces are considered in [4], [8], [14] and [6].

**Theorem 1.** Let $X$ be a compact convex subset of a strictly convex Banach space $B$ with norm $p$. Let $S$ be a left amenable topological semigroup. Then for any (jointly) continuous generalized nonexpansive action of $S$ on $X$, $S$ has a common fixed point in $X$.

**2. Proof of Theorem 1.** Since the given action is generalized nonexpansive there exist nonnegative real numbers $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ such that
\[
\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 = 1
\]
and for all $s \in S$, $x, y \in X$,
\[
p(sx - sy) \leq \alpha_1 p(x - sx) + \alpha_2 p(y - sy) + \alpha_3 p(x - sy) + \alpha_4 p(y - sx) + \alpha_5 p(x - y).
\]
By calculating $(p(sx - sy) + p(sy - sx))/2$ through (2), we may without loss of generality assume that $\alpha_1 = \alpha_2$ and $\alpha_3 = \alpha_4$. Thus $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha = \frac{1}{2}$.

For simplicity, a subset $Y$ of $X$ is said to be invariant if $sy \in Y$ for all $s \in S$, $y \in Y$. By Zorn's lemma, there exists a minimal nonempty closed convex invariant subset $C$ of $X$. Again by Zorn's lemma, there exists a minimal nonempty invariant closed subset $K$ of $C$. We shall first prove that $sK = K$ for all $s \in S$. Let $x_0 \in K$. $x_0$ will now be used to obtain a measure $\lambda$ on the $\sigma$-algebra $B(K)$ of all Borel subsets of $K$. For each $f$ in $B(K)$, let $Tf(s) = f(sx_0)$, $s \in S$. Then $Tf \in \text{LUC}(S)$ [12, proof of Theorem 1]. Since $S$ is left amenable, $\text{LUC}(S)$ has a left invariant mean $\mu$. Let $\lambda = T^*\mu$, where $T^*$ is the adjoint of $T$. Then $\|\lambda\| = \lambda(e) = 1$ and $\lambda(sf) = \lambda(f)$ for all
$f \in C(K)$, $s \in S$, where $sf(y) = f(sy), y \in K$. By the Riesz representation theorem, $\lambda$ can be considered as a measure on $B(K)$ with $\lambda(K) = 1$ and $\lambda(s^{-1}(A)) = \lambda(A)$ for all $s \in S, A \in B(K)$, where $s^{-1}(A) = \{y \in K : sy \in A\}$. Since $K$ is compact, the support $\text{supp} \lambda$ of $\lambda$ is the smallest compact subset $Y$ of $K$ with $\lambda(Y) = 1$. Let $s \in S$. Then from $\lambda(s^{-1}(\text{supp} \lambda)) = \lambda(\text{supp} \lambda) = 1$, we have $s^{-1}(\text{supp} \lambda) \subseteq \text{supp} \lambda$. So $s(\text{supp} \lambda) \subseteq s(s^{-1}(\text{supp} \lambda)) \subseteq \text{supp} \lambda$ and therefore $\text{supp} \lambda$ is invariant. By minimality of $K$, $\text{supp} \lambda = K$. Since $\lambda(sK) = \lambda(s^{-1}(s(K))) \geq \lambda(K) = 1$, $sK \supset \text{supp} \lambda = K$. Hence $sK = K$.

Now note that if $K$ is a singleton, then the point in $K$ is a common fixed point of $S$. So we may assume that $K$ contains at least two points.

Case 1. $\alpha_1 = \alpha_2 = 0$. By Lemma 1 in [3], there exists $z_0 \in C$ such that

$$\sup\{p(z_0 - x) : x \in K\} \leq r$$

for some $r < \delta(K)$ ($= \sup\{p(x - y) : x, y \in K\}$). Let

$$W = \{z \in C : p(z - x) \leq r \text{ for all } x \in K\}.$$ 

Then $z_0 \in W$ and $W$ is a closed convex subset of $C$. To see that $W$ is invariant, let $z \in W, s \in S$. By compactness of $K$, there exists $y_1 \in K$ such that $p(sz - y_1) = \sup\{p(sz - x) : x \in K\}$. Since $sK = K$, $sy_2 = y_1$ for some $y_2 \in K$. Now from (2)

$$p(sz - y_1) = p(sz - sy_2) \leq \alpha_3 p(z - y_1) + \alpha_4 p(sz - y_2) + p(z - y_2) \leq (\alpha_3 + \alpha_4) r + \alpha_4 p(sz - y_1).$$

Since $1 - \alpha_4 \geq \frac{1}{2} > 0$,

$$p(sz - y_1) \leq \frac{\alpha_3 + \alpha_5}{1 - \alpha_4} r = r.$$ 

By the choice of $y_1$, $sz \in W$. Thus $W$ is invariant. By minimality of $C$, $W = C$. Hence by definition of $W$, $\delta(K) \leq r < \delta(K)$, a contradiction.

Case 2. $\alpha_1 = \alpha_2 \neq 0, \alpha_3 = \alpha_4 \neq 0$. Let $s \in S$. Then from $sK = K$ and (2),

$$\delta(K) = \sup\{p(sx - sy) : x, y \in K\} \leq \sup\{\alpha_1 p(x - sx) + \alpha_2 p(y - sy) + \alpha_3 p(x - sy) + \alpha_4 p(y - sx) + \alpha_5 p(x - y) : x, y \in K\} \leq \alpha_1 \sup\{p(x - sx) : x \in K\} + \alpha_2 \sup\{p(y - sy) : y \in K\} + \alpha_3 \sup\{p(x - sy) : x, y \in K\} + \alpha_4 \sup\{p(y - sx) : x, y \in K\} + \alpha_5 \sup\{p(x - y) : x, y \in K\} \leq \alpha_1 \delta(K) + \alpha_2 \delta(K) + \alpha_3 \delta(K) + \alpha_4 \delta(K) + \alpha_5 \delta(K) = \delta(K).$$
Since $\alpha_1 = \alpha_2 \neq 0$, we have from (3),

\begin{equation}
\delta(K) = \sup \{ p(x - sx) : x \in K \}.
\end{equation}

From (4), $sK = K$ and (2),

\begin{align*}
\delta(K) &= \sup \{ p(sx - s^2x) : x \in K \} \\
&\leq \sup \{ \alpha_1 p(x - sx) + \alpha_2 p(sx - s^2x) \\
&\quad + \alpha_3 p(x - s^2x) + \alpha_5 p(x - sx) : x \in K \} \\
&\leq \alpha_1 \delta(K) + \alpha_2 \delta(K) + \alpha_3 \delta(K) + \alpha_5 \delta(K).
\end{align*}

Hence $1 \leq \alpha_1 + \alpha_2 + \alpha_3 + \alpha_5 < 1$, a contradiction.

Case 3. $\alpha_1 = \alpha_2 \neq 0$, $\alpha_3 = \alpha_4 = 0$. Let $s \in S$. By the Schauder-Tychonoff fixed point theorem, $sw = w$ for some $w \in C$. Since $K$ is compact, there exists $y_1 \in K$ such that $p(y_1 - w) = \sup \{ p(x - w) : x \in K \}$. Since $sK = K$, $y_1 = sy_2$ for some $y_2 \in K$. Thus

\begin{align*}
p(w - y_1) &= p(sw - sy_2) \leq \alpha_2 p(y_2 - sy_2) + \alpha_5 p(w - y_2) \\
&\leq \alpha_2 p(y_2 - sy_2) + \alpha_5 p(w - y_1).
\end{align*}

Since $1 - \alpha_5 = \alpha_1 + \alpha_2 > 0$ and $\alpha_2/(1 - \alpha_5) = \frac{1}{2}$,

\begin{align*}
p(w - y_1) &\leq \frac{\alpha_2}{1 - \alpha_5} p(y_2 - sy_2) \leq \frac{1}{2} \delta(K).
\end{align*}

By the choice of $y_1$,

\begin{equation}
p(w - x) \leq \frac{1}{2} \delta(K) \quad \text{for all } x \in K.
\end{equation}

By compactness of $K$, there exist $x_1, x_2 \in K$ such that $\delta(K) = p(x_1 - x_2)$. Now from (5), $\delta(K) = p(x_1 - x_2) \leq p(x_1 - w) + p(w - x_2) \leq \delta(K)$, i.e.

\begin{equation}
p(x_1 - x_2) = p(x_1 - w) + p(w - x_2).
\end{equation}

Since $B$ is strictly convex, we have from (5) and (6), $w = \frac{1}{2}(x_1 + x_2)$. Since $x_1, x_2$ do not depend on $w$, $(x_1 + x_2)/2$ is a common fixed point of $S$.

3. Related results. Let $S$ be a topological semigroup. $S$ is left reversible if the family of all closed right ideals in $S$ has the finite intersection property. When $S$ has the discrete topology, $S$ is left reversible if it is left amenable [5, p. 181]. However, in general, "left reversible" and "left amenable" are two independent conditions on a topological semigroup ([1, p. 516], [7, §4]). By using Lemma 1 in [9], we still have $sK = K$ for all $s \in K$ even if, in Theorem 1, $S$ is left reversible instead of being left amenable. So we have the following result.

**Theorem 2.** Let $X$ be a compact convex subset of a strictly convex Banach space $B$ with norm $p$. Let $S$ be a left reversible topological semigroup.
Then for any (jointly) continuous generalized nonexpansive action of $S$ on $X$, $S$ has a common fixed point in $X$.

Let $S$ be a topological semigroup. A function $f \in C(S)$ is strongly almost periodic if $\{f_s : s \in S\}$ is relatively compact in $C(S)$. Let $AP(S)$ be the family of all strongly almost periodic functions on $S$. Then $AP(S)$ is a Banach subalgebra of $LUC(S)$ which contains all constant real-valued functions on $S$ and is translation invariant [2]. An action of $S$ on a compact subset $X$ of a Banach space is equicontinuous if $S$ is equicontinuous when it is considered as a family of functions of $X$ into $X$. Now by using Lemma 3.1 in [10] and by modifying the proof of Theorem 1 in an obvious way, we have the following result.

**Theorem 3.** Let $X$ be a compact subset of a Banach space $B$ with norm $p$. Let $S$ be a topological semigroup such that $AP(S)$ has a left invariant mean. Then for any equicontinuous and jointly continuous action of $S$ on $X$, $S$ has a common fixed point in $X$.

We would like to point out here that by modifying the definitions and proofs in an obvious way, one can prove Theorems 1–3 for the case when $B$ is a Hausdorff locally convex topological space with its topology induced by a given family of pseudonorms on $B$.

**REFERENCES**

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