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A SCHWARZ LEMMA FOR CANONICAL ALGEBRAIC MANIFOLDS

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ABSTRACT. It was proved that a holomorphic mapping with a fixed point between a canonical algebraic manifold is biholomorphic if and only if the absolute value of the determinant of its differential at the fixed point is 1.

In this note we prove the following theorem.

THEOREM. *Let V_n be a compact complex manifold such that the canonical line bundle $K=\bigwedge^n T_V^*$ is very ample ($n=\dim_C V$). Let x_0 be a point of V and $f: V\to V$ be a holomorphic mapping such that $f(x_0)=x_0$. We denote by $df_0: T_0(V)\to T_0(V)$ the differential of f at x_0 where $T_0(V)$ denotes the tangent space of V at x_0. Then*

(1) $$|\det df_0| \leqq 1,$$

(2) *df_0 is the identity transformation if and only if f is the identity transformation of V, and*

(3) $|\det df_0| = 1$ *if and only if f is a biholomorphic mapping.*

For bounded domain and hyperbolic manifold, the above theorem is due to H. Cartan and S. Kobayashi [**2**].

Before we start proving the theorem we recall a few facts, proved by Griffiths [**1**], about holomorphic mappings into a canonical algebraic manifold (i.e. $\bigwedge^n T_V^*$ is very ample).

PROPOSITION 1. *Let V be a complete canonical algebraic manifold of dimension n and $w_0, \cdots, w_N$ be a basis for the vector space of holomorphic n-forms on V. Suppose that $\{f_k^\#\}$ is an arbitrary sequence of holomorphic mappings of the unit ball B about 0 of C^n into V. Then*

(1) *There exists a subsequence $\{f_k\}$ of $\{f_k^\#\}$ such that the pull-backs $f_k^* w_i$ converge uniformly on compact sets to a holomorphic n-form φ_i ($i=0, \cdots, N$).*

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(2) *If* $|\det df_k(0)| \geqq 1$, *the meromorphic mapping* $f: B \to V$ *given by the homogeneous coordinates* $f = [\varphi_0, \cdots, \varphi_N]$ *is holomorphic in a neighborhood* U *of* 0 *and the sequence* $\{f_k\}$ *converges uniformly on compact subsets of* U *to* f.

PROPOSITION 2. *Let* V *be a complete canonical algebraic manifold of dimension* n *and* B *the unit ball about* 0 *in* C^n. *Let* $f: B \to V$ *be an arbitrary holomorphic mapping with* $|\det df(0)| \geqq 1$. *Then there exists an absolute constant* $r > 0$ *such that there is a univalent ball* $\Delta(f(0), r)$ *of radius* r *about* $f(0)$ *for* $f: B \to V$.

A univalent ball $\Delta(x_0, r)$ for $f: B \to V$ is by definition a disc $\Delta(x_0, r)$ on V of radius r about x_0 such that f maps some open set U in B biholomorphically onto $\Delta(x_0, r)$. The above two propositions are proved in [1].

Now we prove our theorem. Let $f: V \to V$ be a holomorphic mapping such that $f(x_0) = x_0$. Assume (1) of the theorem is false, i.e. $|\det df_0| = a > 1$. For each positive integer k, the mapping $f^k = f \circ \cdots \circ f$ (k-times) satisfies $|\det df_0^k| = a^k$. By Proposition 1, there is a neighborhood U of x_0 such that, when restricted to U, a subsequence of $\{f^k\}$ converges uniformly on compact subsets of U to a holomorphic mapping $g: U \to V$. Since a^k diverges to infinity as k goes to infinity, we arrive at a contradiction. This proves (1).

We denote by $d^m f_0$ all partial derivatives of f of order m at x_0. We will show that if df_0 is the identity transformation of $T_0(V)$, then $d^m f_0 = 0$ for $m \geqq 2$. Let m be the least integer $\geqq 2$ such that $d^m f_0 \neq 0$. Then $d^m(f^k)_0 = k d^m f_0 \neq 0$ for all positive integers k. As k goes to infinity, $d^m(f^k)_0$ also goes to infinity in contradiction to the fact that a subsequence of $\{f^k\}$ converges uniformly to a holomorphic mapping in a neighborhood of x_0. This proves (2).

If f is a biholomorphic mapping, then the inverse f^{-1} is a well-defined holomorphic mapping of V into V such that $f^{-1}(x_0) = x_0$. Therefore if $|\det df_0| < 1$, $|\det df_0^{-1}| > 1$. By (1) of the theorem we must have $|\det df_0| = 1$ if f is a biholomorphic mapping.

Assume $|\det df_0| = 1$. The mapping $f^k = f \circ \cdots \circ f$ (k-times) has a subsequence which converges uniformly on compact subsets to a holomorphic mapping g is a neighborhood of x_0. Let λ denote an eigenvalue of df_0. Then df_0^k has an eigenvalue λ^k. If $|\lambda| > 1$, then λ^k goes to infinity as k goes to infinity, which is a contradiction. Therefore the eigenvalues of df_0 have absolute value less than or equal to one. As $|\det df_0| = 1$, $|\lambda| = 1$. Now put df_0 in Jordan canonical form. We claim that df_0 is then in diagonal form, and the diagonal entries are all of the form $e^{i\theta}$. If it is not, it must

have a diagonal block of the form

$$\begin{bmatrix} \lambda & 1 & & & & \\ & \lambda & 1 & & 0 & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \cdot \\ & 0 & & & \cdot & 1 \\ & & & & & \lambda \end{bmatrix}, \qquad |\lambda| = 1.$$

The corresponding block of df_0^k is then of the form

$$\begin{bmatrix} \lambda^k & k\lambda^{k-1} & & & & \\ & \cdot & & & * & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & 0 & & & \cdot & k\lambda^{k-1} \\ & & & & & \lambda^k \end{bmatrix}.$$

It follows that the entries $k\lambda^{k-1}$ immediately above the diagonal of df_0^k diverge to infinity as k goes to infinity. This is again a contradiction to the convergence of a subsequence of $\{f^k\}$ in a neighborhood of x_0.

Since df_0 is a diagonal matrix whose entries have absolute value 1, there is a subsequence $\{df_0^{k(i)}\}$ of $\{df_0^k\}$ such that $\{df_0^{k(i)}\}$ converges to the identity matrix. By Proposition 1, there is a subsequence of $\{f^{k(i)}\}$, denoted again by $\{f^{k(i)}\}$, which converges uniformly to a holomorphic mapping h on a compact neighborhood of x_0. Since $\{df_0^{k(i)}\}$ converges to dh_0, dh_0 is the identity matrix. Then h is the identity mapping of a neighborhood of x_0, which can be proved in a similar way as (2) of the theorem.

Let W be the largest open subset of V with the property that some subsequence of $\{f^{k(i)}\}$ converges to the identity transformation of W uniformly on compact subsets. Without loss of generality we may assume that $\{f^{k(i)}\}$ converges to the identity transformation of W uniformly on compact subsets. Let $p \in \overline{W}$ and U be a neighborhood of p with compact closure. By Proposition 1, there is a subsequence of $\{f^{k(i)}\}$, denoted again by $\{f^{k(i)}\}$, such that the pull-backs $f^{k(i)*}(w_j)$ $(j=0, \cdots, N)$ converge uniformly on compact subsets to holomorphic n-forms φ_j, where $w_0, \cdots, w_N$ are a basis for the vector space of holomorphic n-forms on V. Then the mapping given by the homogeneous coordinates $f=[\varphi_0, \cdots, \varphi_N]$ is the identity mapping on $W \cap U$ and is meromorphic on U. It follows that

$f=[\varphi_0, \cdots, \varphi_N]$ must define the identity mapping on U. This proves that W is closed and hence $W=V$.

By change of notation we may assume that $\{f^{k(i)}\}$ converges to the identity transformation of V uniformly on compact subsets. To show f is one-to-one, let $f(x)=f(y)$. Then $f^{k(i)}(x)=f^{k(i)}(y)$ and as $k(i)$ goes to infinity, we get $x=y$. This shows f is one-to-one. Finally we show f is onto. Suppose it is not. Then there is a point p in $V-f(V)$. We also have $p \in V-f^{k(i)}(V)$ for all $k(i)$. Since $\{df^{k(i)}\}$ converges to the identity matrix, $|\det df_p^{k(i)}| \geqq 1-\varepsilon>0$ for sufficiently large $k(i)$'s. By Proposition 2, there is a univalent ball for $f^{k(i)}$ about $f^{k(i)}(p)$ of radius $r>0$ for sufficiently large $k(i)$'s. Since $f^{k(i)}(p)$ converges to p as $k(i)$ goes to infinity, p belongs to all of these univalent balls for all sufficiently large $k(i)$'s. Hence $p \in f^{k(i)}(V)$ for all sufficiently large $k(i)$'s contradicting $p \in V-f^{k(i)}(V)$. This proves the theorem.

References

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