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RESIDUALLY F_p-GROUPS, FOR MANY PRIMES p, ARE ORDERABLE

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ABSTRACT. It is proved that if G is a residually finite p-group, for infinitely many primes p, then G can be linearly ordered.

Let F_p denote the class of finite p-groups. It is well known that free groups and free polynilpotent groups are residually F_p for all primes p. It is also known that such groups can be linearly ordered ([**2**, p. 49], [**1**, Theorem 4]). The purpose of this note is to prove the following general result on the orderability of residually F_p-groups.

THEOREM. *If G is locally a residually F_p-group for infinitely many primes p, then G can be linearly ordered.*

To prove the Theorem we shall need the following result.

LEMMA. *Let α_{ij} $(i=1,\cdots,2^n; j=1,\cdots,n)$ be nonnegative integers such that $\sum_{j=1}^n \alpha_{ij}>0$ for all i. Let p be a prime such that $p\geqq(2\alpha)^{2^n}$ where $\alpha=\max\{\alpha_{ij}\}$. Then in any solution set to the system of 2^n equations $\sum_{j=1}^n \varepsilon_{ij}\alpha_{ij}x_j\equiv 0 \pmod p$ associated with signs $\varepsilon_{ij}=\pm 1$, $x_j\equiv 0 \pmod p$ for some j.*

PROOF. We use induction on n. The result is trivially true when $n=1$. Assume the result holds for $n<m$ and let $n=m$. Let $d=2^{m-1}$ and for any integer r let $r'=d+r$. By renumbering the 2^m equations, if necessary, assume that $\varepsilon_{i1}=1$, $\varepsilon_{i'1}=-1$ and $\varepsilon_{ij}=\varepsilon_{i'j}$ for all $i\in\{1,\cdots,d\}$ and $j>1$. We produce a system of d equations in $m-1$ unknowns satisfying the hypotheses of the Lemma as follows. For any $r\in\{1,\cdots,d\}$ we have

$$\text{I}\qquad \alpha_{r1}x_1+\sum_{j=2}^{m}\varepsilon_{rj}\alpha_{rj}x_j\equiv 0 \pmod p,$$

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and

$$\text{II} \qquad -\alpha_{r'1}x_1 + \sum_{j=2}^{m} \varepsilon_{rj}\alpha_{r'j}x_j \equiv 0 \pmod{p}.$$

As the rth equation of the new system pick: I if $\alpha_{r1}=0$; II if $\alpha_{r1}\neq 0$ but $\alpha_{r'1}=0$; $\sum_{j=2}^{m} \varepsilon_{rj}(\alpha_{r1}\alpha_{r'j}+\alpha_{r'1}\alpha_{rj})x_j \equiv 0 \pmod{p}$ if $\alpha_{r1}\neq 0$ and $\alpha_{r'1}\neq 0$. Notice that for any r if $\sum_{j=2}^{m} \alpha_{rj}=0$ then $x_1\equiv 0\ (p)$ and the Lemma is proved. Thus we may assume that at least one coefficient of the rth equation of the new system is positive. Let β be the maximum of the coefficients of the new system of equations. Then $\beta \leqq (2\alpha^2)$ so that $p \geqq (2\alpha)^{2^m} = (4\alpha^2)^{2^{m-1}} \geqq (2\beta)^{2^{m-1}}$. Thus the hypotheses of the Lemma are satisfied for this system and, by induction, $x_j \equiv 0 \pmod{p}$ for some j.

PROOF OF THE THEOREM. We use a characterization of ordered groups given by Fuchs [**2**, p. 34]. If x is an element of a group G and α a non-negative integer, we write $[x]^\alpha$ to denote a product of α conjugates of x in G.

Suppose, if possible, that the Theorem is false. Then there exist elements $x_i \neq e$ $(i=1, \cdots, n)$ in G such that

$$(1) \qquad [x_1^{\varepsilon_{i1}}]^{\alpha_{i1}} \cdots [x_n^{\varepsilon_{in}}]^{\alpha_{in}} = e$$

for 2^n values of i associated with signs $\varepsilon_{ij}=\pm 1$. Moreover $\sum_{j=1}^{n} \alpha_{ij}>0$ for all i. Assume that n is the least such integer. Let $\alpha=\max\{\alpha_{ij}\}$. We can assume that G is generated by the finite set of elements involved in the equations given by (1), so that G is residually F_p for infinitely many p. Choose a prime $p>(2\alpha)^{2^n}$ such that G is residually F_p for this prime. Let $X=\langle x_1^G, \cdots, x_n^G\rangle$, the normal subgroup of G generated by $x_1, \cdots, x_n$. There exists $K \lhd G$ maximal subject to $G/K \in F_p$, $X \nleqq K$.

Let θ be the homomorphism $G \to G/K = G^*$, and let $\theta(x_i)=a_i$, $\theta(X)=A$. Then $G^* \in F_p$, $A \neq \{e\}$, and, by our choice of K, A is a minimal normal subgroup of G^*. Thus A is cyclic of order p in the centre of G^*. By re-ordering the suffixes, if necessary, assume that $A=\langle a_1\rangle$, $a_j=a_1^{\beta_j}$ where $\beta_1=1$, $0<\beta_j<p$ for $j=2, \cdots, m$ and $\beta_j=0$ for $j>m$. The 2^n equations given by (1) now reduce to the following.

$$a_1^{(\sum_{j=1}^{m} \varepsilon_{ij}\alpha_{ij}\beta_j)} = e, \qquad j = 1, \cdots, m,\ i = 1, \cdots, 2^n.$$

If $m<n$, then from the 2^n equations above choose 2^m equations associated with the 2^m choices of signs $\varepsilon_{ij}=\pm 1$ $(j=1, \cdots, m)$ in such a way that for each $i \in \{1, \cdots, 2^m\}$ at least one $\alpha_{ij}\neq 0$ in this set of 2^m equations. This is possible by our choice of n. The above equations now reduce to

$$\sum_{j=1}^{m} \varepsilon_{ij}\alpha_{ij}\beta_j \equiv 0 \pmod{p}, \qquad i = 1, \cdots, 2^m.$$

The above equations satisfy the hypotheses of the Lemma and hence $\beta_j \equiv 0 \pmod{p}$ for some $j \in \{1, \cdots, m\}$. This is a contradiction and the Theorem is proved.

References

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