THE APPROXIMATION PROPERTY DOES NOT IMPLY
THE BOUNDED APPROXIMATION PROPERTY

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Abstract. There is a Banach space which has the approximation property but fails the bounded approximation property. The space can be chosen to have separable conjugate, hence there is a nonnuclear operator on the space which has nuclear adjoint. This latter result solves a problem of Grothendieck [2].

I. Introduction. Let $(X, \| \cdot \|)$ be a Banach space. We show that if there is a constant $\lambda$ so that $(X, \| \cdot \|)$ has the $\lambda$-metric approximation property ($\lambda$-m.a.p., in short) for each equivalent norm $\| \cdot \|$ on $X$, then $X^*$ has the bounded approximation property (b.a.p., in short). This result is used to construct an example of a Banach space which possesses the approximation property (a.p.) but fails the b.a.p.

For $\epsilon, \lambda$ positive constants, we say that $X$ has the $(\epsilon, \lambda)$-m.a.p. provided that, for each finite dimensional subspace $Z$ of $X$ and each $\delta > 0$, there is a finite rank operator $T$ on $X$ so that $\| T \| \leq \lambda + \delta$ and $\| Tz - z \| \leq (\epsilon + \delta) \| z \|$ for each $z \in Z$. An intermediate step in our construction is that if $X$ has the $(\epsilon, \lambda)$-m.a.p. for some $\epsilon, 0 < \epsilon < 1$, then $X$ has the $\lambda(1 - \epsilon)^{-1}$-m.a.p.

We use the standard notation in Banach space theory. Let us only recall the types of approximation conditions a Banach space $X$ may satisfy. $X$ has the a.p. if for each compact subset $K$ of $X$ and $\epsilon > 0$, there is a finite rank operator (= bounded, linear operator) $T$ on $X$ so that $\| Tk - k \| \leq \epsilon$ for each $k \in K$. If always $T$ can be chosen with $\| T \| \leq \lambda$ then $X$ is said to have the $\lambda$-m.a.p. A space which has the $\lambda$-m.a.p. for some $\lambda$ is said to have the b.a.p. For equivalent formulations of these definitions (which we use without further reference) the reader is referred to [2] and [4].

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II. Implications among approximation conditions. Given a Banach space \((X, \| \cdot \|)\), let \(\mathscr{A}\) be the family of equivalent norms, \(\| \cdot \|\), on \(X\) whose dual norms on \(X^*\) are of the form \(\| x^* \| = \| x^* \| + M d(x^*, Z)\). Here \(M\) ranges over positive constants, \(Z\) ranges over finite dimensional subspaces of \(X^*\), and \(d(x^*, Z) = \inf \{ \| x^* - z \| : z \in Z \}\) is the \(\| \cdot \|\)-distance of \(x^*\) to \(Z\). Since finite dimensional subspaces of \(X^*\) are weak* closed, it is evident that each such norm on \(X^*\) is the dual of an equivalent norm on \(X\).

**Proposition 1.** Suppose that \((X, \| \cdot \|)\) has the \(\lambda\)-m.a.p. for each norm, \(\| \cdot \|\), in \(\mathscr{A}\). Let \(0 < \varepsilon < 1\). Then \((X^*, \| \cdot \|)\) has the \((\varepsilon, \lambda[1 + 2\varepsilon^{-1}\lambda])\)-m.a.p. 

**Proof.** Suppose that \(Z\) is a finite dimensional subspace of \(X^*\). Let \(\beta > \lambda\) and \(\delta > 0\). Define \(\| \cdot \|\) on \(X^*\) by \(\| x^* \| = \| x^* \| + 2\varepsilon^{-1}\beta d(x^*, Z)\).

Pick a finite dimensional subspace \(Y\) of \(X\) such that for each \(z \in Z\), \(\| z \| \leq (1 + \delta) \sup \{ \| y \| : y \in Y, \| y \| \leq 1 \}\). Since \((X, \| \cdot \|)\) has the \(\lambda\)-m.a.p., there is a finite rank operator \(T\) on \(X\) so that \(Ty = y\) for \(y \in Y\) and \(\| T \| \leq \beta\).

We have, for \(x^* \in X^*\),

\[
\| T^* x^* \| + 2\varepsilon^{-1}\beta d(T^* x^*, Z) \leq \beta \| x^* \| + 2\varepsilon^{-1}\beta d(x^*, Z).
\]

Hence \(\| T^* x^* \| \leq \beta (1 + 2\varepsilon^{-1}\beta) \| x^* \|\) whence \(\| T \| \leq \beta (1 + 2\varepsilon^{-1}\beta)\). Now for \(z \in Z\), \(2\varepsilon^{-1}\beta d(T^* z, Z) \leq \| z \|\), so there exists \(w \in Z\) satisfying \(\| T^* z - w \| \leq \frac{1}{2}\varepsilon \| z \|\). But for \(y \in Y\), \((T^* z)y = z(Ty) = z(y)\), and thus \(\sup \{ \| z(y) - w(y) \| : y \in Y, \| y \| \leq 1 \} \leq \frac{1}{2}\varepsilon \| z \|\). Therefore \(\| z - w \| \leq \frac{1}{2}\varepsilon (1 + \delta) \| z \|\), from which it follows that

\[
\| T^* z - z \| \leq \left[ \frac{1}{2} \varepsilon (1 + \delta) + \frac{1}{2} \varepsilon \right] \| z \| \leq (1 + \delta) \varepsilon \| z \|.
\]

Since \(\delta > 0\), \(\beta > \lambda\) are arbitrary, the conclusion follows.

**Proposition 2.** Suppose \((X, \| \cdot \|)\) has the \((\varepsilon, \lambda)\)-m.a.p. with \(\varepsilon < 1\). Then \(X\) has the \((1 - \varepsilon)^{-1} \lambda\)-m.a.p.

**Proof.** We thank Professor W. J. Davis for the proof given here. Davis’ proof is rather more revealing than proofs discovered by us.

Suppose \(Z\) is a finite dimensional subspace of \(X\). Let \(0 < \varepsilon < \delta < 1\) and \(\beta > \lambda\).

Construct by induction finite rank operators \(T_n\) on \(X\) so that

\[
\| T_1 z - z \| \leq \delta \| z \| \quad \text{for } z \in Z, \quad \| T_{n+1} x - x \| \leq \delta \| x \|
\]

for \(x \in \text{span } Z \cup T_n X \cup T_{n-1} X \cup \cdots \cup T_1 X\), and \(\| T_n \| \leq \beta\).

Define \(S_n\) by \(I - S_n = (I - T_n)(I - T_{n-1}) \cdots (I - T_1)\). Then for \(z \in Z\), \(\| (I - S_n) z \| \leq \delta^n \| z \|\). Also,

\[
S_n = (I - T_n)(I - T_{n-1}) \cdots (I - T_3)T_1 + (I - T_n)(I - T_{n-1}) \cdots (I - T_3)T_2 + \cdots + (I - T_n)T_{n-1} + T_n,
\]

so that \(\| S_n \| \leq \delta^n \beta + \cdots + \delta \beta + \beta < (1 - \delta)^{-1}\beta\).
Hence $X$ has the $(\tau, (1-\delta)^{-\beta})$-m.a.p. for each $\tau > 0$, $\delta > \varepsilon$ and $\beta > \lambda$, whence $X$ has the $(1-\varepsilon)^{-\lambda}$-m.a.p.

 Setting $\varepsilon = \frac{1}{3}$ in the above two propositions yields:

**Theorem 1.** If, for each $| \cdot |$ in $\mathcal{A}$, $(X, | \cdot |)$ has the $\lambda$-m.a.p., then $X^*$ has the $2\lambda(1+4\lambda)$-m.a.p.

**Remark 1.** If $X^*$ has the $\lambda$-m.a.p. then for each $| \cdot |$ in $\mathcal{A}$, $(X^*, | \cdot |)$ has the $\lambda$-m.a.p. (hence also $(X, | \cdot |)$ has the $\lambda$-m.a.p.). For if $|x^*| = \|x^*\| + M d(x^*, Z)$, $Y$ is a finite dimensional subspace of $X^*$, and $\varepsilon > 0$, then there is a finite rank operator $T$ on $X^*$ so that $\|T\| \leq \lambda + \varepsilon$ and $Tx^* = x^*$ for $x^* \in \text{span} Y \cup Z$. Since $T$ is the identity on $Z$, $|T| \leq \|T\| \leq \lambda + \varepsilon$.

**Remark 2.** It is known [3, Theorem 4] that if $(X, | \cdot |)$ has the $1$-m.a.p. for each $| \cdot |$ in $\mathcal{A}$, then $X^*$ has the $1$-m.a.p. We do not know whether a similar result is true with “1” replaced by “$\lambda$.” It may even be true that if $X^*$ has the b.a.p., then $X^*$ also has the $1$-m.a.p. This is the case when $X^*$ is separable [4, Remark 4.11].

**Example.** There is a Banach space which has the a.p. but fails the b.a.p.

**Proof.** Of course, we need the important result of Enflo [1] that there is a Banach space which fails the a.p. Lindenstrauss [5] (see [3] for a specific example) had shown that a consequence of this is the existence of a Banach space $X$ which possesses the $1$-m.a.p., but whose conjugate fails the a.p. By Theorem 1, there is a sequence $(| \cdot |_n)$ of equivalent norms on this $X$ so that $(X, | \cdot |_n)$ fails the $n$-m.a.p. Thus $(\Sigma (X, | \cdot |_n))_{l_2}$ fails the b.a.p. but possesses the a.p.

Note that $(\Sigma (X, | \cdot |_n))_{l_2}$ can be chosen to have separable conjugate, since Lindenstrauss’ construction can yield an $X$ with $X^*$ separable.

**III. Nonnuclear operators with nuclear adjoints.** The example constructed in §II justifies the following proposition.

**Proposition 3.** If a Banach space $X$ has the a.p. but fails the b.a.p., and $X^*$ is separable, then there is a nonnuclear operator $T$ on $X$ such that $T^*$ is nuclear.

Since the proof is an almost immediate consequence of results of [2], we give only some indications.

Let $N(X)$ denote the space of nuclear operators on $X$ [2, Definition 4] and $L_0(X)$ the space of finite rank operators on $X$. Since the weak*-continuous nuclear operators form a closed subset of $N(X^*)$, it is enough to show that $\{ T^*: T \in N(X) \}$ is not closed in $N(X^*)$.

Consider the natural mappings $X^* \otimes X^* \rightarrow N(X)^* \rightarrow [L_0(X)]^* \rightarrow N(X^*)$. Here $\chi(T)S = \text{trace } ST$ and $\psi(T)S = \text{trace } (TS^*)$. 

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Observe that

(i) \( \varphi \) is an isometry onto, because \( X \) has the a.p. (cf. [2, Proposition 35, \( A \Rightarrow B_1 \)]).

(ii) \( \chi \varphi \) is not an isomorphic embedding, for otherwise (cf. [2, Proposition 39, proof of \( B_1 \Rightarrow A_1 \)]) \( X \) would have the b.a.p.

(iii) \( \psi \) is an isometry onto. For given \( F \in L_0(X)^* \), consider the factorization (cf. [2, Proposition 27, (a)\( \Rightarrow \)(d)]) \( X^* \rightarrow L_\infty \rightarrow L_1 \rightarrow X^* \) of the operator induced on \( X^* \) by \( F \). \( X^* \) is separable, so the Dunford-Pettis theorem yields (cf. [2, Lemma 9]) that \( L_\infty \rightarrow L_1 \rightarrow X^* \) is nuclear.

Since (i), (ii), and (iii) imply that the range of \( \psi^{-1} \chi \) is not closed, it only remains to observe that \( \psi^{-1} \chi(T) = T^* \) for each \( T \in N(X) \).

REFERENCES


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