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# NEGLIGIBILITY IN NONLOCALLY CONVEX SPACES[1]

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ABSTRACT. A negligibility theorem is established in a linear topological space without assuming the existence of a convex body or linearly bounded open set.

1. A set $A$ in a topological space $X$ is negligible if $X$ is homeomorphic to $X\backslash A$. Negligibility investigations in linear topological spaces include [1], [2], [3], [4], [6], and [7]. Using shrinkable neighborhoods of Ives [5] and Klee [8], we adapt methods of Bessaga and Klee to prove the following theorem.

(1.1) THEOREM. *Suppose* $(X, \tau_1)$ *is a linear topological space admitting a linear topology* $\tau_2 \subset \tau_1$ *such that* $(X, \tau_2)$ *is metrizable and incomplete, and that* $K$ *is* $\tau_2$*-compact,* $U$ *is* $\tau_2$*-open with* $[0, 1]K \subset U$. *Then there is a* $\tau_1$*-homeomorphism* $h: X \to X\backslash K$ *with* $h|X\backslash U = \mathrm{Id}$.

(1.2) COROLLARY. *Suppose* $(X, \tau_1)$ *is a metrizable complete linear topological space, and* $\tau_2$ *is metrizable and strictly weaker than* $\tau_1$. *If* $K$, $U$ *are as in* (1.1), *the conclusion holds.*

(1.3) COROLLARY. *Let* $M$ *be the space of a.e. finite Lebesgue measurable functions,* $S$ *the simple functions, and* $C$ *the continuous functions, all on* $[0, 1]$. *If* $S \subset X \subsetneq M$ *or* $C \subset X \subsetneq M$, *and if convergence in* $X$ *implies convergence in measure, then subsets of* $X$ *which are compact in* $M$ (*with the topology of convergence in measure*) *are negligible in* $X$.

(1.4) COROLLARY. *Suppose the hypotheses of* (2.1) *hold, except* $U \in \tau_1$, $K$ *is* $\tau_1$*-compact, and* $\tau_2$ *contains a linearly bounded set. Then the conclusion of the theorem holds.*

It follows from (1.3) that $M$-compact sets are negligible in $L^p$, $p>0$. A case of Anderson's result on $\alpha$-spaces [1] follows from the theorem.

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An $\alpha$-space is an infinite dimensional linear topological space with a Schauder basis $\{b_n\}$, continuous coordinates, and an open neighborhood $U$ of 0 such that $b_n \notin U$ for each $n$. If $(X, \tau)$ is an $\alpha$-space, then $X$ may be regarded as a linear subspace of $s$. If $X \neq s$, then $\tau_1 | X \subset \tau$, where $\tau_1$ is the topology of coordinatewise convergence, and $X$ is dense in $(s, \tau_1)$, since if $x \in s$, $(x_1, x_2, \cdots, x_n, 0, 0, \cdots) \in X$, and $\to x$. Thus $(X, \tau_1)$ is incomplete, and the theorem applies to show $\tau_1$-compact sets negligible. If $X = s$, and $X$ is metrizable, again $\tau_1 \subset \tau$ and $b_n \to O(\tau_1)$, but $b_n \not\to O(\tau)$ since $b_n \notin U$. By the open mapping theorem, $(X, \tau_1)$ is incomplete, and again $\tau_1$-compact sets are negligible in $(X, \tau)$.

If $U$ is a set in a linear topological space, and $p \in \operatorname{Int} U$, then $U$ is shrinkable at $p$ if $[0, 1)(U-p)^{-} \subset \operatorname{Int}(U-p)$. Notice that each ray $r$ from $p$ meets Bd $U$ at most once, and that $r \cap \bar{U}$ is closed and connected. If $U$ is shrinkable at $p$, the gauge functional $\gamma_U(x, p)$ is defined by $x = p + \gamma_U(x, p)(\pi_U(x, p) - p)$ in case ray $\boldsymbol{px}$ meets Bd $U$ in $\pi_U(x, p)$. If $\boldsymbol{px} \subset U$ or $x = p$, then $\gamma_U(x, p) = 0$. Ives [**5**] has shown that $\gamma_U(x, p)$ is continuous in $x$. It follows that $\pi_U(x, p)$, as a function of $x$, is continuous on its domain. According to Klee [**8**], each Hausdorff linear topological space has a basis at 0 of open sets, shrinkable at 0. A set $A$ in a linear space is star-shaped at $a \in A$ if $tx + (1-t)a \in A$ whenever $x \in A$, $t \in [0, 1)$. In the following $[A]$ will denote the convex hull of $A$.

(1.5) LEMMA. *Suppose $X$ is a metrizable linear topological space and $K, W \subset X$ with $K$ compact, $W$ open and shrinkable at 0. If $x \in X$, $k \in K$ and $r > 0$, then $\{k + \lambda(x-k) \mid \lambda \geqq 0\} \subset K + rW$ if and only if $\{\lambda(x-k) \mid \lambda \geqq 0\} \subset W$.*

PROOF. Assume the first inclusion, and take $\lambda \geqq 0$. For each $n$, $k + nr(x-k) \in K + rW$, so that $k + nr(x-k) = k_n + rw_n$ with $k_n \in K$, $w_n \in W$, and $w_n = (1/r)(k - k_n) + n(x-k) \in W$. $\{k_n\}$ has a convergent subsequence $\{k_{n_l}\}$. If $l$ is large,

$$(\lambda + 1)/n_l < 1, \quad \text{and} \quad ((\lambda + 1)/n_l)((1/r)(k - k_{n_l}) + n_l(x - k)) \in W.$$

Letting $l \to \infty$, $(\lambda + 1)(x-k) \in \overline{W}$. Therefore $\lambda(x-k) \in W$. The converse is clear.

Proofs of the following lemmas are straightforward and omitted. Lemma (1.9) is due to Klee [**8**].

(1.6) LEMMA. *Let $Y$ be a linear subspace of a linear topological space $X$, $y \in Y$, and $U$ open, shrinkable at $y$. Then $U \cap Y$ is shrinkable at $y$ in $Y$.*

(1.7) LEMMA. *Let $(X, \tau_1)$ be a linear topological space, and $\tau_2$ a weaker linear topology for $X$. If $U \in \tau_2$ is $\tau_2$-shrinkable at 0, then $U$ is $\tau_1$-shrinkable at 0, $\bar{U}^{\tau_1} = \bar{U}^{\tau_2}$, and* $\operatorname{Bd}_{\tau_1} U = \operatorname{Bd}_{\tau_2} U$.

(1.8) LEMMA. *If $X$ is a linear space, and $A$, $B\subset X$ with $A$ star-shaped at $x\in A$, then $C=\bigcup\{\lambda A+(1-\lambda)B|\lambda\in[0,1]\}$ is star-shaped at $x$.*

(1.9) LEMMA. *If $U$ is open, shrinkable at 0, and $K$ is compact, star-shaped at $k\in K$, then $U+K$ is shrinkable at $k$. In particular, if $K$ is convex, $U+K$ is shrinkable at each point of $K$.*

We now prove (1.1). Let $\tilde{X}$ be a linear metric completion of $(X,\tau_2)$, and $\{\tilde{W}_n\}$ a basis of open sets, shrinkable at 0, with $\tilde{W}_n^-\subset W_{n-1}$. $U=\tilde{U}\cap X$ for some open $\tilde{U}$. Using $[0,1]K\subset U$ and the compactness of $K$, we can find $\tilde{x}\in\tilde{X}\backslash X$ such that $\bigcup\{\lambda\tilde{x}+(1-\lambda)K|\lambda\in[0,1]\}\subset\tilde{U}$. Let $y_n\to\tilde{x}$ with $y_n\in U$. Again the compactness of $K$ implies the existence of $n_1$ such that

$$\bigcup\{\lambda[\tilde{x},y_{n_1}]+(1-\lambda)K\,|\,\lambda\in[0,1]\}\subset\tilde{U}.$$

Let $x_1=y_{n_1}$ and $\tilde{K}_1=\bigcup\{\lambda[\tilde{x},x_1]+(1-\lambda)K|\lambda\in[0,1]\}$, a compact subset of $\tilde{U}$. There exists $l_1$ such that $(\tilde{K}_1+3\tilde{W}_{l_1})^-\subset\tilde{U}$. Let

$$\begin{aligned} \tilde{A}_1 &= [\tilde{x},x_1]+\tilde{W}_{l_1}, & A_1 &= \tilde{A}_1\cap X,\\ \tilde{B}_1 &= [\tilde{x},x_1]+2\tilde{W}_{l_1}, & B_1 &= \tilde{B}_1\cap X,\\ \tilde{C}_1 &= \tilde{K}_1+2\tilde{W}_{l_1}, & C_1 &= \tilde{C}_1\cap X,\\ \tilde{D}_1 &= \tilde{K}_1+3\tilde{W}_{l_1}, & D_1 &= \tilde{D}_1\cap X. \end{aligned}$$

By (1.8), (1.9), (1.6) and (1.7), $A_1$, $B_1$, $C_1$ and $D_1$ are $\tau_1$-shrinkable at $x_1$.

$$\begin{aligned} \bar{A}_1 &= \tilde{A}_1^-\cap X=([\tilde{x},x_1]+\tilde{W}_{l_1})^-\cap X\\ &= ([\tilde{x},x_1]+\tilde{W}_{l_1}^-)\cap X\subset([\tilde{x},x_1]+2\tilde{W}_{l_1})\cap X=B_1. \end{aligned}$$

Similarly we get $\bar{A}_1\subset B_1\subset C_1\subset\bar{C}_1\subset D_1$. The statements $\{x_1+\lambda(x-x_1)|\lambda\geqq 0\}\subset A_1$, $\subset B_1$, $\subset C_1$, $\subset D_1$ are equivalent by (1.5). Now we define $h_1:X\to X$. $h_1|\bar{A}_1\cup(X\backslash D_1)=\mathrm{Id}$. $h_1|\bar{B}_1\backslash A_1:\bar{B}_1\backslash A_1\to\bar{C}_1\backslash A_1$ is defined as follows. If $y\in\bar{B}_1\backslash A_1$, then $\pi_{A_1}(y,x_1)$ is defined and, by the above remark, so are $\pi_{B_1}(y,x_1)$, $\pi_{C_1}(y,x_1)$ and $\pi_{D_1}(y,x_1)$. If $r$ is a ray from $x_1$, intersecting Bd $A_1$, $h_1$ maps $(\bar{B}_1\backslash A_1)\cap r$ linearly onto $(\bar{C}_1\backslash A_1)\cap r$. Thus,

$$h_1(y)=\pi_{A_1}(y,x_1)+\frac{\gamma_{A_1}(y,x_1)-1}{\gamma_{A_1}(\pi_{B_1}(y,x_1),x_1)-1}(\pi_{C_1}(y,x_1)-\pi_{A_1}(y,x_1)),$$

is a continuous function. $h_1|\bar{B}_1\backslash A_1$ has an inverse of the same form, so it is a homeomorphism. Similarly, we define $h_1|\bar{D}_1\backslash B_1:\bar{D}_1\backslash B_1\to\bar{D}_1\backslash C_1$. $h_1$ is then a homeomorphism. Another compactness argument shows there exists $n_2>n_1$ such that

$$[\tilde{x},y_{n_2}]\subset\tilde{A}_1\quad\text{and}\quad\bigcup\{\lambda[\tilde{x},y_{n_2}]+(1-\lambda)K\,|\,\lambda\in[0,1]\}\subset\tilde{C}_1.$$

Let $x_2=y_{n_2}$, $\tilde{K}_2=\bigcup\{\lambda[\tilde{x}, x_2]+(1-\lambda)K \mid \lambda\in[0,1]\}$. Then $[\tilde{x}, x_2]\subset\tilde{A}_1$, $\tilde{K}_2\subset\tilde{C}_1$ and there exists $l_2>l_1$ such that $([\tilde{x}, x_2]+2\tilde{W}_{l_2})^{-}\subset\tilde{A}_1$, $(\tilde{K}_2+3\tilde{W}_{l_2})^{-}\subset\tilde{C}_1$. Let

$$\begin{aligned}
\tilde{A}_2 &= [\tilde{x}, x_2] + \tilde{W}_{l_2}, & A_2 &= \tilde{A}_2 \cap X,\\
\tilde{B}_2 &= [\tilde{x}, x_2] + 2\tilde{W}_{l_2}, & B_2 &= \tilde{B}_2 \cap X,\\
\tilde{C}_2 &= \tilde{K}_2 + 2\tilde{W}_{l_2}, & C_2 &= \tilde{C}_2 \cap X,\\
\tilde{D}_2 &= \tilde{K}_2 + 3\tilde{W}_{l_2}, & D_2 &= \tilde{D}_2 \cap X.
\end{aligned}$$

As before, $A_2$, $B_2$, $C_2$, $D_2$ are $\tau_1$-shrinkable at $x_2$, $\bar{A}_2\subset B_2\subset C_2\subset\bar{C}_2\subset D_2$, and the statements $\{x_2+\lambda(x-x_2) \mid \lambda\geqq 0\}\subset A_2$, $\subset B_2$, $\subset C_2$, $\subset D_2$ are equivalent. Also $\bar{B}_2\subset A_1$ and $\bar{D}_2\subset C_1$. Define $h_2: X\to X$ so that

$$h_2 \mid \bar{A}_2 \cup (X\backslash D_2) = \mathrm{Id}, \qquad h_2 \mid \bar{B}_2\backslash A_2 : \bar{B}_2\backslash A_2 \to \bar{C}_2\backslash A_2,$$
$$h_2 \mid \bar{D}_2\backslash B_2 : \bar{D}_2\backslash B_2 \to \bar{D}_2\backslash C_2.$$

Note $h_2 \mid X\backslash C_1=\mathrm{Id}$. Continue, obtaining sets

$$\begin{array}{ccccccccccc}
D_1 & \supset & C_1 & \supset & B_1 & \supset & A_1 & & & & \\
 & & \cup & & & & \cup & & & & \\
 & & D_2 & \supset & C_2 & \supset & B_2 & \supset & A_2 & & \\
 & & & & \cup & & & & \cup & & \\
 & & & & D_3 & \supset & C_3 & \supset & B_3 & \supset & A_3 \\
 & & & & & & & & \cdot & & \\
 & & & & & & & & & \cdot & \\
 & & & & & & & & & & \cdot
\end{array}$$

and homeomorphisms $h_n: X\to X$ with

$$\begin{aligned}
h_n \mid \bar{A}_n \cup (X\backslash D_n) &= \mathrm{Id},\\
h_n \mid \bar{B}_n\backslash A_n &: \bar{B}_n\backslash A_n \to \bar{C}_n\backslash A_n,\\
h_n \mid \bar{D}_n\backslash B_n &: \bar{D}_n\backslash B_n \to \bar{D}_n\backslash C_n,\\
h_n \mid X\backslash C_{n-1} &= \mathrm{Id}.
\end{aligned}$$

We claim $\bigcap \bar{A}_n=\bigcap \bar{B}_n=\varphi$. Since $\bar{A}_n\subset\bar{B}_n\subset A_{n-1}\subset B_{n-1}$, it is sufficient to show $\bigcap B_n=\varphi$. If $y\in\bigcap B_n$, then $y\in X$, and $y\in[\tilde{x}, x_n]+2\tilde{W}_{l_n}$, so that $y=\tilde{x}$, a contradiction. Also $\bigcap \bar{C}_n=\bigcap \bar{D}_n=K$. Suppose $y\in\bigcap D_n$. Then $y\in X$ and $y\in\tilde{K}_n+3\tilde{W}_{l_n}$, so that $y=\lambda\tilde{x}+(1-\lambda)k$. If $\lambda\neq 0$, $\tilde{x}=(1/\lambda)(y-(1-\lambda)k)\in X$, a contradiction. Therefore $y=k\in K$. Thus $K\subset\bigcap \bar{C}_n\subset\bigcap\bar{D}_n\subset\bigcap D_n\subset K$, since $\bar{D}_n\subset C_{n-1}\subset D_{n-1}$. If we trace the motion of a point $x\in X$ under the successive homeomorphisms $h_1, h_2, \cdots$, we see $x\notin\bar{B}_n$ for some $n$, and $h_{n+k}\cdots h_2h_1 \mid X\backslash\bar{B}_n=h_n\cdots h_2h_1 \mid X\backslash\bar{B}_n$. Thus we may define a homeomorphism $h$ on $X$ by $h(x)=\cdots h_2h_1(x)$. It is not hard to see $h$ is onto $X\backslash K$. Since $X\backslash U\subset X\backslash D_1$, evidently $h|X\backslash U=\text{Id}$.

Corollary (1.2) is proved by noting $\text{Id}:(X, \tau_1)\to(X, \tau_2)$ is not an open map, so the open mapping theorem implies $\tau_2$ is incomplete. For Corollary (1.3) note that $X$ with the topology of convergence in measure is incomplete, since it is dense and a proper subspace of $M$. Finally, we prove Corollary (1.4).

Using the hypothesis we can find $U_1\in\tau_1$, $V_1\in\tau_2$, both linearly bounded and $\tau_1$-shrinkable at zero, with $[0, 1]K\subset U_1\subset U$ and $U_1\subset V_1$. Since $[0, 1]K$ is $\tau_1$-compact, $[0, 1]K\subset rU_1$ for some $r\in(0, 1)$. There is a $\tau_1$-homeomorphism $j:X\to X$ such that

$$j\mid r\bar{U}_1\cup(X\backslash 2V_1)=\text{Id},\qquad j\mid \bar{U}_1\backslash rU_1:\bar{U}_1\backslash rU_1\to\bar{V}_1\backslash rU_1,$$

and

$$j\mid 2\bar{V}_1\backslash U_1:2\bar{V}_1\backslash U_1\to 2\bar{V}_1\backslash V_1.$$

By the theorem, there exists a $\tau_1$-homeomorphism $h:X\to X\backslash K$ such that $h|X\backslash V_1=\text{Id}$. Then $j^{-1}hj:X\to X\backslash K$ is a $\tau_1$-homeomorphism fixed on $X\backslash U$.

2. In the proof of (1.1), the homeomorphisms Id, $h_1$, $h_2h_1$, $h_3h_2h_1,\cdots$ may be regarded as successive stages of an isotopy whose final homeomorphism is $h:X\to X\backslash K$. There are obvious ways to fill the gaps, but the details are tedious. The statement of the isotopy theorem below is patterned after Klee's [**6**], and the corollary extends his theorem to an arbitrary normed linear space. A full development of these results will appear elsewhere.

(2.1) THEOREM. *Suppose $(X, \tau_1)$ is a linear topological space admitting a metrizable incomplete linear topology $\tau_2\subset\tau_1$. If $U\in\tau_2$ and $K$ is $\tau_2$-compact with $[0, 1]K\subset U$, then there exists a $\tau_1$-embedding $H:X\times[0, 1]\to X\times[0, 1]$ such that if $f_t(x)=p_1H(x, t)$ (projection on the first coordinate) for $t\in[0, 1]$, then $\{f_t\}$ has the following properties.*

1. *$f_t:X\to X$ is a $\tau_1$-homeomorphism for each $t\in[0, 1)$.*
2. *$f_0=\text{Id}$.*
3. *$f_1:X\to X\backslash K$ is a $\tau_1$-homeomorphism.*
4. *For each $t\in[0, 1]$, $f_t|X\backslash U=\text{Id}$.*
5. *$\lim_{t\to 1}f_1f_t^{-1}=\text{Id}_X(\tau_1)$ and $\lim_{t\to 1}f_tf_1^{-1}=\text{Id}_{X\backslash K}(\tau_1)$ with the convergence uniform on each $\tau_2$-compact set.*

(2.2) COROLLARY. *Suppose the hypotheses of the theorem hold, except that $U\in\tau_1$, $K$ is $\tau_1$-compact and $[0, 1]K\subset U$. Suppose also $\tau_2$ contains a linearly bounded set. Then the conclusions of the theorem hold (with limits uniform on $\tau_1$-compact sets).*

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