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AN IMBEDDING THEOREM FOR SEPARABLE ALGEBRAS

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ABSTRACT. Let S/R be a ring extension, where S is a commutative ring. If S/R is strongly separable then it can be imbedded in a weakly Galois extension of R in the sense of [7, Definition 3.1].

Throughout this paper, all rings will be assumed commutative. Moreover, R will mean a ring with identity element 1, and all ring extensions of R will be assumed with identity element coinciding with the identity element of R. A ring extension S/R is called strongly separable if S is a separable R-algebra which is projective as an R-module (and so, S is a finitely generated R-module by [6, Villamayor's theorem]). Moreover, a ring S will be called connected if S has no proper idempotents. In [1], Auslander and Goldman proved that if S/R is a ring extension which is strongly separable and S is a free R-module then S/R is imbedded in a Galois extension of R. Moreover, in [4], Janusz proved that if S/R is a ring extension which is strongly separable and S is connected then S/R is imbedded in a Galois extension N of R such that N is connected. Now, in [7], Villamayor and Zelinsky gave the notion of weakly Galois extensions. A ring extension N/R is weakly Galois if and only if there exists a finite number of orthogonal idempotents $e_1, \cdots, e_t$ in R such that $\sum_{i=1}^{t} e_i=1$ and the extensions Ne_i/Re_i are Galois ([7, 3.15]). In [5], Magid proved that some types of rings have separable closures, and strongly separable extensions of the rings are determined by the Galois groupoids of the closures.

In this note, we shall prove the following theorem by using the results of [7].

THEOREM. *Every ring extension S/R which is strongly separable can be imbedded in a weakly Galois extension of R.*

To prove the theorem we need several lemmas. Now, let R be a connected ring, and S/R a ring extension which is strongly separable. Then

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we have a direct decomposition into connected rings:

(1) $$S = S_1 \oplus \cdots \oplus S_m.$$

Then each S_i is a projective separable R-algebra. By the result of [**4**, §1], there exists a Galois extension of R:

(2) $$R[T_{1,1}, \cdots, T_{1,n_1}, \cdots, T_{m,1}, \cdots, T_{m,n_m}]$$

such that it is connected, each $T_{i,j}$ is a R-algebra which is isomorphic to S_i, and $n_i \leqq \text{rank}_R(S)$ in the sense of [**2**, Définition 2.5.2]. This Galois extension is unique up to isomorphism, and it is isomorphic to a unique R-subalgebra of the separable closure of R in the sense of [**4**, Definition 3]. Hence this will be denoted by $C(S/R)$.

First, we shall prove the following

LEMMA 1. *Let R be a connected ring, and S/R a ring extension which is strongly separable. Then $C(S/R) \otimes_R S$ is a Galois extension of R.*

PROOF. As is easily seen, we have an R-algebra isomorphism

$$C(S/R) \otimes_R S \cong C(S/R) \oplus \cdots \oplus C(S/R).$$

Then, by making use of the same methods as in the proof of [**7**, 3.15], we see that $C(S/R) \otimes_R S$ is a Galois extension of R.

Now, we denote by $\otimes^s S$ the tensor product of s copies of S over R. Next, we shall prove the following

LEMMA 2. *Let R be a connected ring, S/R a ring extension which is strongly separable, and S an R-module generated by n elements. Then, for any integer $s \geqq n^2$, $\otimes^s S$ contains an idempotent f such that $(\otimes^s S)f$ is an R-algebra isomorphic to $C(S/R)$.*

PROOF. We consider a decomposition of S as in the preceding (1):

$$S = S_1 \oplus \cdots \oplus S_m.$$

Let F be the set of pairs (i, j) such that $i=1, \cdots, m$; if $i<m$ then $j=1, \cdots, n$, and if $i=m$ then $j=1, \cdots, t=s-(m-1)n$. Since $t \geqq n$, the preceding (2) may be rewritten as

$$C(S/R) = R[T_{1,1}, \cdots, T_{i,j}, \cdots, T_{m,t}] \qquad ((i,j) \in F)$$

where $S_i \cong T_{i,j}$ (as R-algebras) for every $(i,j) \in F$. We set $S_{i,j}=S$, $(i,j) \in F$. Then, we have R-algebra epimorphisms

$$f_{i,j}: S_{i,j} \to T_{i,j}$$

and

$$f = \otimes f_{i,j}: \otimes^s S = \otimes S_{i,j} \to C(S/R).$$

Hence, by [4, Lemma 1.6], there exists an idempotent f in $\otimes^s S$ such that $(\otimes^s S)f \cong C(S/R)$ (as R-algebras). This completes the proof.

As in [7, 2.1], let $B(R)$ be the Boolean ring consisting of all idempotents of R, and let Spec $B(R)$ be the Boolean spectrum of R which is the Stone space consisting of all prime ideals of $B(R)$, where the family of the subsets $U_e = \{y \in \operatorname{Spec} B(R); e \in y\}$ $(e \in B(R))$ forms a base of the open subsets of $B(R)$. For any $x \in \operatorname{Spec} B(R)$, we denote by R_x the ring of residue classes of R modulo the ideal $\sum_{e \in x} Re$. Then R_x is a connected ring ([7, 2.13]). Moreover, for any R-module M, we denote by M_x the tensor product $R_x \otimes_R M$, and for any element $a \in M$, we denote by a_x the image of a under the canonical homomorphism $M \to M_x$. Finally, we shall prove the following lemma which contains the assertion of our theorem.

LEMMA 3. *Let S/R be a ring extension which is strongly separable, and S an R-module generated by n elements. Then for any integer $s \geqq n^2$, $\otimes^s S$ contains an idempotent f such that $(\otimes^s S)f \otimes_R S$ is a weakly Galois extension of $f \otimes R$ and $f \otimes S$ is an R-algebra isomorphic to S.*

PROOF. Let $x \in \operatorname{Spec} B(R)$. Then S_x is a ring extension of R_x which is strongly separable, and is an R_x-module generated by n elements. Let s be an integer $\geqq n^2$. Clearly $(\otimes^s S)_x \cong \otimes^s S_x$ (the tensor product over R_x). Hence by Lemma 2, $\otimes^s S_x$ contains an idempotent g_x such that $(\otimes^s S_x)g_x \cong C(S_x/R_x)$ (as R_x-algebras). We set $T = (\otimes^s S) \otimes_R S$. Then

$$T_x = (\otimes^s S_x) \otimes_{R_x} S_x = (\otimes^s S_x)g_x \otimes S_x \oplus (\otimes^s S_x)(1_x - g_x) \otimes S_x.$$

Since S_x is a projective R_x-module, we have $(\otimes^s S_x)g_x \otimes S_x \cong (\otimes^s S_x)g_x \otimes_{R_x} S_x$. Hence by Lemma 1, $(\otimes^s S_x)g_x \otimes S_x$ is a Galois extension of $R_x(g_x \otimes 1_x)$ with a Galois group G_x, and so, by [3, Theorem 1.3], there exist elements $a_1, \cdots, a_k; b_1, \cdots, b_k$ in $(\otimes^s S)g \otimes S$ such that

$$\sum_{i=1}^{k} a_{ix}\sigma_x(b_{ix}) = \delta_{1,\sigma_x}(g_x \otimes 1_x) \quad \text{for all } \sigma_x \in G_x,$$

where $\delta_{1,1} = 1$, and if $1 \neq \sigma_x$, $\delta_{1,\sigma_x} = 0$. As is easily seen, G_x is extended to a group of R_x-algebra automorphisms of T_x. Hence by [7, 2.14], G_x can be lifted to a set of R-algebra automorphisms of T, which will be denoted by G. Then, by using the results of [2, Théorème 2.5.1] and [7, 2.9], we can choose a neighborhood U_e $(= \{y \in \operatorname{Spec} B(R); e \in y\})$ of x such that for all $y \in U_e$,

$$g_y^2 = g_y, \qquad \operatorname{rank}_{R_y}((\otimes^s S_y)g_y \otimes S_y) = \operatorname{rank}_{R_x}((\otimes^s S_x)g_x \otimes S_x),$$

$$\sum_{i=1}^{k} a_{iy}\sigma(b_i)_y = \delta_{1,\sigma}(g_y \otimes 1_y) \quad \text{for all } \sigma \in G,$$

and the restriction of G to $(\otimes^s S_y)g_y \otimes S_y$ is a group which is isomorphic to G_x under the correspondence: the restriction of σ to $(\otimes^s S_y)g_y \otimes S_y \to$ the restriction of σ to $(\otimes^s S_x)g_x \otimes S_x$, where $\sigma \in G$. Now, we set $u=1-e$. Then

$$(gu)^2 = gu, \qquad \operatorname{rank}_{Ru}((\otimes^s S)gu \otimes Su) = \operatorname{rank}_{R_x}((\otimes^s S_x)g_x \otimes S_x),$$
$$\sum_{i=1}^{k} a_i u\sigma(b_i u) = \delta_{1,\sigma}(gu \otimes u) \quad \text{for all } \sigma \in G,$$

and the restriction of G to $(\otimes^s S)gu \otimes Su$ is a group of Ru-algebra automorphisms, whose order coincides with that of G_x and with $\operatorname{rank}_{Ru}((\otimes^s S)gu \otimes Su)$. Hence $(\otimes^s S)gu \otimes Su$ is a Galois extension of $Rgu \otimes u$. Since $(\otimes^s S)gu \otimes Su \cong (\otimes^s S)gu \otimes_{Ru} Su$, and gu is Ru-free, $gu \otimes Su$ is an Ru-algebra isomorphic to Su. By the compactness of Spec $B(R)$, there exist pairwise orthogonal idempotents $u_1, \cdots, u_t$ in R and elements $g_1, \cdots, g_t$ in $\otimes^s S$ such that $\sum_{i=1}^t u_i^2 = 1$, for each i, $(g_i u_i)^2 = g_i u_i$, $(\otimes^s S)g_i u_i \otimes S u_i$ is a Galois extension of $Rg_i u_i \otimes u_i$, and $g_i u_i \otimes S u_i$ is an Ru_i-algebra isomorphic to Su_i. We set $f = \sum_{i=1}^t g_i u_i$. Then

$$\sum_{i=1}^{t} (\otimes^s S)g_i u_i \otimes S u_i = (\otimes^s S)f \otimes S \cong (\otimes^s S)f \otimes_R S,$$

which is a weakly Galois extension of $Rf \otimes 1 = f \otimes R$, and $f \otimes S$ is an R-algebra isomorphic to S. This completes the proof.

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