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THE FINITENESS OF I WHEN $R[X]/I$ IS R-PROJECTIVE

J. W. BREWER AND P. R. MONTGOMERY

ABSTRACT. This paper is concerned with the relationship between $R[X]/I$ being a projective R-module and I being a finitely generated ideal of $R[X]$. It is shown that if $R[X]/I$ is R-free, then $I=fR[X]$, f a monic polynomial of $R[X]$. Also, $R[X]/I$ is a finitely generated projective R-module if and only if $R[X]/I$ is a finitely generated R-module and $I=fR[X]$ for some $f \in R[X]$. When $R[X]/I$ is projective, I is a finitely generated ideal if and only if I is a principal ideal. Finally, an example is given to show that $R[X]/I$ can be projective without I being finitely generated.

R will always denote a commutative ring with identity 1, X an indeterminate over R and I an ideal of $R[X]$. We remark that $R[X]/I$ is a finitely generated R-module if and only if I contains a monic polynomial.

If A is an ideal of $R[X]$, then by the *content* of A, written $c(A)$, we mean the ideal of R generated by the elements of R which occur as a coefficient of some element of A. It is convenient here to call to mind the result of Heinzer-Ohm-Rush [**3**, Corollary 1.6] that $R[X]/I$ is R-flat if and only if I is locally (at each maximal ideal of $R[X]$) a principal ideal and $R/c(I)$ is R-flat.

We shall write $A \subset B$ to mean that A is contained in, but not equal to, B.

Finally, we would like to acknowledge several beneficial conversations which we had with Douglas Costa during the writing of the paper.

The key tool of our paper is a result of Miyashita [**5**, Theorem 1.3] which characterizes I in case $R[X]/I$ is a finitely generated projective R-module. To state the result we need the notion of a *quasi-monic* polynomial of $R[X]$. $f \in R[X]$ is said to be quasi-monic if there exists a finite collection $e_0, e_1, \cdots, e_n$ of pairwise orthogonal idempotents such that $\sum_{i=0}^{n} e_i = 1$ and, for $0 \leqq i \leqq n$, $e_i f$ is a monic polynomial in the ring $e_i R[X]$. It is easy to see that, if f is quasi-monic, we can choose $e_0, e_1, \cdots, e_n$ in such a way that $\text{degree}(e_0 f) > \text{degree}(e_1 f) > \cdots > \text{degree}(e_n f)$. Miyashita's theorem says that, if I is a nonzero ideal of $R[X]$, then $R[X]/I$ is a finitely

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generated projective R-module if and only if $I=fR[X]$, where f is a quasi-monic polynomial of $R[X]$. By means of this result we can show that if $R[X]/I$ is R-free then $I=fR[X]$, f monic. This result should be known but apparently is not in the literature. The case of quasi-local R appears as Proposition 4.8 of [7]. We utilize a portion of their argument in our proof.

In the proof and in the sequel we shall have occasion to use the notion of what we call a *pure* ideal. A is a pure ideal of R if R/A is a flat R-module. (See [2, Proposition 3.4] for several equivalent formulations; the terminology there being $*$-ideal. Pure ideal seems to us more appealing, since it calls to mind the fact that $0\to A\to R$ is a pure sequence of R-modules—that is, $0\to A\otimes_R E\to R\otimes_R E$ is exact for each R-module E.) Finally, we recall from [7, Corollary 1.3] the fact that if $R[X]/I$ is R-flat, then $c(I)$ is a pure ideal of R.

THEOREM 1. *If I is a nonzero ideal of $R[X]$, the following conditions are equivalent*:

(1) $I=fR[X]$, *f a monic polynomial in $R[X]$.*

(2) *$R[X]/I$ is a finitely generated free R-module.*

(3) *$R[X]/I$ is a free R-module.*

PROOF. That (1) implies (2) is well known and that (2) implies (3) is obvious. Thus, we need only prove that (3) implies (1). Set $R[X]/I=P$. P R-free implies that $c(I)$ is a pure ideal of R and, since $I\neq(0)$, $c(I)\neq(0)$. Therefore, there exists a maximal ideal M of R such that $c(I)R_M=R_M$. For this M, $MI\neq I$ for equality would yield that $M[X]\supseteq MI$ and hence that $c(I)\subseteq M$. Consider the following exact sequences:

$$0\to I\to R[X]\to P\to 0,\qquad 0\to I/MI\xrightarrow{\phi}(R/M)[X]\to P/MP\to 0.$$

Note that the exactness of the second is a consequence of the purity of the first. Now $\operatorname{Ker}\phi=(M[X]\cap I)/MI=(0)$ and so $M[X]\cap I=MI$. Thus, $I/MI=I/(M[X]\cap I)=(I+M[X])/M[X]$ is an ideal of $R[X]/M[X]\simeq(R/M)[X]$. Moreover, $I\supset MI$ and so P/MP is a finitely generated free (R/M)-module. By [4, p. 418], P is a finitely generated free R-module. Thus, by Miyashita's theorem $I=fR[X]$, where f is quasi-monic with respect to $e_0, e_1, \cdots, e_n$ and $\operatorname{degree}(e_0f)>\cdots>\operatorname{degree}(e_nf)$. If $n>0$, then there exist maximal ideals M_0 and M_1 of R so that $e_0\notin M_0$, $e_1\notin M_1$. (If an idempotent belongs to each maximal ideal, it is zero.) Moreover, since a maximal ideal avoids exactly one of the idempotents, we have that $e_1\in M_0$ and $e_0\in M_1$. It follows that f, when regarded as a polynomial in $R_{M_0}[X]$, is such that $\operatorname{degree}(e_0f)>\operatorname{degree}(e_1f)=$degree of f regarded as a polynomial in $R_{M_1}[X]$. But then $\operatorname{rank}(P_{M_0})>\operatorname{rank}(P_{M_1})$ and the rank of P is locally constant by [4, p. 418]. We conclude that $n=0$ and f is monic. Q.E.D.

We now turn our attention to the case when $R[X]/I$ is a finitely generated projective R-module.

THEOREM 2. *The following conditions are equivalent*:

(1) *$R[X]/I$ is a finitely generated projective R-module.*

(2) *$I=fR[X]$, f a quasi-monic polynomial of $R[X]$.*

(3) *$I=fR[X]$, where for each maximal ideal M of R, the leading coefficient of the image of f in $R_M[X]$ is a unit of R_M.*

(4) *$R[X]/I$ is projective and $c(I)=R$.*

(5) *$R[X]/I$ is finitely generated and $I=fR[X]$ for some $f\in R[X]$.*

PROOF. (1)$\Rightarrow$(2). This is [**5**, Theorem 3].

(2)$\Rightarrow$(3). Let $f=\sum_{i=0}^{n} a_iX^i$ be quasi-monic with respect to $e_0, e_1, \cdots, e_r$ where degree $(e_0f)>\cdots>$degree(e_rf) and let M be a maximal ideal of R. Then there exists a unique integer i such that $e_i\notin M$. By choice of e_0, $e_1, \cdots, e_r$ there exists a nonnegative integer k such that $e_ia_k=e_i$ and $e_ia_j=0$, $k+1\leqq j\leqq n$. Hence, the image of f is monic in $R_M[X]$.

(3)$\Rightarrow$(4). The hypothesis on f clearly implies that $c(I)=R$. Thus, let M be a maximal ideal of R. If $f=\sum_{i=0}^{n} a_iX^i$, by hypothesis there exist a nonnegative integer k and elements $t_i\notin M$ such that $t_ia_i=0$, $k+1\leqq i\leqq n$ and $a_k\notin M$. If $t=(\prod_{k+1}^{n} t_i)\cdot a_k$, then $t\notin M$; and if we denote by S_t the localization of the ring S with respect to the powers of the element t, the image of f has a unit leading coefficient in $R_t[X]$ and $(R[X]/I)_t\simeq R_t[X]/fR_t[X]$ is a finitely generated free R_t-module. By [**1**, p. 138], $R[X]/I$ is a projective R-module.

(4)$\Rightarrow$(5). Let M be a maximal ideal of R. Since $c(I)=R$, $IR_M[X]\neq(0)$ and since $R[X]/I$ is projective, $R_M[X]/IR_M[X]$ is R_M-free. Moreover, $c(IR_M)=R_M$ and hence by Theorem 1, $IR_M[X]$ is generated by a monic polynomial. It follows easily that there exists $g\in I$ such that the leading coefficient of g does not belong to M. Let this coefficient be t. Then $IR_t[X]$ contains a monic polynomial, namely the image of g. Thus, $(R[X]/I)_t$ is a finitely generated R_t-module and, by [**1**, p. 137], $R[X]/I$ is a finitely generated R-module. That I is principal now follows from the implication (1) implies (2) above.

(5)$\Rightarrow$(1). Since $R[X]/I$ is finitely generated, I contains a monic and, consequently, $c(I)=R$. By the Heinzer-Ohm-Rush theorem, $R[X]/I$ is R-flat. (In fact, one can appeal to [**6**, p. 439].) Suppose that R has no nonzero nilpotent elements. If M is a maximal ideal of R, since $(R[X]/I)_M$ is a finitely generated free R_M-module, by Theorem 1, $IR_M[X]=gR_M[X]$ for some $g\in I$ such that the leading coefficient of g does not belong to M. Therefore, $f^*h^*=g^*$ and $f^*=g^*h_1^*$ for polynomials h, $h_1\in R[X]$ (p^* denotes the image in $R_M[X]$ of a polynomial p of $R[X]$). Thus, $h^*h_1^*=1^*$ and so the constant term of h_1^* is a unit and all other coefficients are nilpotent. But R_M has no nonzero nilpotent elements, since R has none. It follows that h_1^* is a unit in R_M and that the leading coefficient of f^* is a unit of R_M. By (3) implies (4) above, $R[X]/I$ is projective.

We now pass to the general case. Given $0 \to I \to R[X] \to R[X]/I \to 0$ exact with I principal and $R[X]/I$ finitely generated, denote by P the module $R[X]/I$ and by N the nilradical of R. Then $0 \to I/NI \to (R/N)[X] \to P/NP \to 0$ is exact, since P is R-flat. By the reduced case, P/NP is projective. So we need only see that the projectivity descends. Thus, let E be a finitely generated flat R-module such that $E \otimes_R (R/N)$ is (R/N)-projective. By [**1**, p. 138] there exist $t_1^*, \cdots, t_m^* \in R/N$ such that $(E \otimes_R (R/N))_{t_i^*}$ is $(R/N)_{t_i^*}$-free for $1 \leqq i \leqq m$ and $(t_1^*, \cdots, t_m^*) = R/N$. Then $(t_1, \cdots, t_m) = R$ and

$$\begin{aligned} E_{t_i} \otimes_{R_{t_i}} (R_{t_i}/N_{t_i}) &\cong E_{t_i} \otimes_{R_{t_i}} (R/N)_{t_i} \cong E_{t_i} \otimes_{R_{t_i}} (R/N)_{t_i^*} \\ &\cong E \otimes_R (R_{t_i} \otimes_R (R/N)_{t_i^*}) \cong (E \otimes_R (R/N))_{t_i^*}. \end{aligned}$$

As argued, for example, in the proof of Proposition 4 of [**8**], if F is a finitely generated flat R-module such that $F \otimes_R (R/N)$ is a free (R/N)-module, F is already a free R-module. Thus, in our case, E_{t_i} is a finitely generated free R_{t_i}-module for $1 \leqq i \leqq m$ and so by [**1**, p. 138] E is a finitely generated projective R-module. Q.E.D.

Ohm and Rush [7, Corollary 4.9] proved that (4) implies (5) and as they correctly observe, this is the appropriate generalization of Vasconselos' result [**10**]. Also, Raphael [**8**] proved that (5) implies (1) under the assumption that R is semiconnected. Raphael's Proposition 4, the main theorem of [**8**], is an immediate consequence of our Theorem 2, since a quasi-monic polynomial in a connected ring is monic.[1] The fact that Theorem 2 holds in an arbitrary ring in no way detracts from the above authors' contributions; quite the contrary. They did not have Miyashita's result at their disposal.

The following result is a rather easy consequence of the Heinzer-Ohm-Rush theorem, but it is instructive to compare it with the other results of this type.

PROPOSITION 1. *The following conditions are equivalent*:

(1) *$R[X]/I$ is a finitely generated flat R-module.*

(2) *For each maximal ideal M of R, $IR_M[X]$ is principal generated by a monic polynomial of $R_M[X]$.*

PROOF. If M is a maximal ideal of R and if $R[X]/I$ is a finitely generated flat R-module, then $(R[X]/I)_M \simeq R_M[X]/IR_M[X]$ is a finitely generated

[1] We would like to thank the referee for bringing to our attention the paper of Raphael.

free R_M-module and therefore $IR_M[X]$ is principal generated by a monic polynomial of $R_M[X]$.

Conversely, if Q is a maximal ideal of $R[X]$ and if $Q_0=Q\cap R$, then there is a maximal ideal M of R with $Q_0\subseteq M$. $IR_M[X]$ is principal, whence $I(R[X])_Q$ is principal and so by [**3**, Corollary 1.6], $R[X]/I$ is flat. One sees readily that if N is any maximal ideal of R, there is a polynomial $f\in I$ whose leading coefficient t does not belong to N and $IR_N[X]=fR_N[X]$. This being true for M, choose $t\in R$, $t\notin M$ with the above properties. Then $(c(f))_M=R_M$ and, moreover, f has a unit leading coefficient in $R_t[X]$ and $f\in IR_t[X]$. Thus, $(R[X]/I)_t\simeq R_t[X]/IR_t[X]$, being an R_t-homomorphic image of the finitely generated R_t-module $R_t[X]/fR_t[X]$, is itself finitely generated. Thus, by [**1**, p. 137], $R[X]/I$ is finitely generated. Q.E.D.

We remark that $R[X]/I$ being a finitely generated flat R-module is not equivalent to $R[X]/I$ being a finitely generated projective R-module. If the notions were equivalent, then for each pure ideal A of a ring R, $R[X]/(A, X)\simeq R/A$ would be a projective R-module and, hence, A would be principal generated by an idempotent. But pure ideals need not be principal. In fact, the rings R for which each pure ideal is principal have been much studied and termed $A(0)$-rings in [**2**]. This is a large class of rings including all integral domains, all Noetherian rings and all rings having only a finite number of maximal ideals. Not all rings are $A(0)$ however; $C([0, 1])$ affords a relatively easy example of a ring which is not $A(0)$.

We have been unable to characterize those ideals I for which $R[X]/I$ is projective, but there is strong evidence to indicate that a satisfactory characterization does not exist. When I is finitely generated, something can be said. Namely,

THEOREM 3. *If $R[X]/I$ is R-projective, then the following conditions are equivalent*:

(1) $I=fR[X]$.
(2) *I is finitely generated.*
(3) *$c(I)$ is finitely generated.*
(4) *$c(I)=eR$, e idempotent.*

Moreover, if any, and hence each, of these conditions is satisfied, $I=fR[X]$ where there exist pairwise orthogonal idempotents $v_0, v_1, \cdots, v_r$ of R such that v_if is monic in $v_iR[X]$ and if $v=\sum_{i=0}^{r} v_i$, then $(1-v)f=0$.

PROOF. That (1) implies (2) and (2) implies (3) is obvious, and that (3) implies (4) follows since $c(I)$ is a pure ideal of R. Thus, we need only prove

(4)$\Rightarrow$(1). $R=eR\oplus e_1R$, where $e_1=1-e$. Thus, $R[X]=eR[X]\oplus e_1R[X]$ with $I=IeR[X]$ since $Ie_1=0$. Therefore, the eR-content of $IeR[X]$ is eR

and $R[X]/I \simeq (eR[X]/IeR[X]) \oplus e_1R[X]$. Moreover,

$$(R[X]/I) \oplus_R (R/e_1R) \simeq (R[X]/I)/(e_1R[X] + I/I) \simeq (R[X])/(e_1R[X] \oplus I)$$
$$= (e_1R[X] \oplus eR[X])/(e_1R[X] \oplus I) \simeq eR[X]/IeR[X]$$

is eR-projective since $R[X]/I$ is R-projective. By Theorem 2, $I = IeR[X]$ is principal as an ideal of $eR[X]$, and, *a fortiori*, as an ideal of $R[X]$.

To complete the proof of the theorem, we have seen that if (4) is satisfied, then $I = IeR[X] = feR[X]$, where f is quasi-monic in $eR[X]$. Thus, there exist pairwise orthogonal idempotents $eu_0, eu_1, \cdots, eu_n$ of eR with $e = \sum_{i=0}^n eu_i$, eu_if is monic in $eu_iR[X]$, $0 \leqq i \leqq n$, and degree$(eu_0f) > \cdots >$ degree(eu_nf). But $eu_0, eu_1, \cdots, eu_n$ are pairwise orthogonal idempotents of R and $(1-e)f = 0$. Q.E.D.

Taking the usual approach to such a situation we define a polynomial $g \in R[X]$ to be *almost quasi-monic* if there exist pairwise orthogonal idempotents $v_0, v_1, \cdots, v_n$ of R with v_if monic in $v_iR[X]$, $0 \leqq i \leqq n$, and if $v = \sum_{i=0}^n v_i$, then $(1-v)f = 0$. Notice that this is precisely the statement that there exists an idempotent $v \in R$ such that vf is quasi-monic in $vR[X]$.

Thus, in this terminology we have proved that if $R[X]/I$ is R-projective, then I is finitely generated if and only if I is principal generated by an almost quasi-monic polynomial. We can also prove the following

PROPOSITION 2. *Suppose that $I = fR[X]$ with f almost quasi-monic. Then $R[X]/I$ is R-projective.*

PROOF. We may assume that f is almost quasi-monic with respect to $e_0, e_1, \cdots, e_n$ and that degree$(e_0f) > \cdots >$ degree(e_nf). Let $e = \sum_{i=0}^n e_i$. Then e is idempotent, $c(f) = eR$ and

$$R[X]/I = (eR[X]/IeR[X]) \oplus ((1-e)R[X]/I(1-e)R[X])$$
$$= (eR[X]/IeR[X]) \oplus ((1-e)R[X]).$$

Now, $(1-e)R[X]$ is a direct summand of the free R-module $R[X]$ and hence is R-projective. Moreover, $IeR[X] = feR[X]$ and f is quasi-monic in $eR[X]$. Thus, $eR[X]/IeR[X]$ is a finitely generated projective eR-module and consequently is a projective R-module. It follows that $R[X]/I$ is R-projective. Q.E.D.

Therefore, we have seen that if $I = fR[X]$, where f is almost quasi-monic, then $R[X]/I$ is R-projective and not necessarily finitely generated. But it can happen that $R[X]/I$ is R-projective without I being finitely generated.

PROPOSITION 3. *Let R be a ring containing a sequence of idempotents $\{e_i\}_{i=0}^{\infty}$ satisfying the following conditions*:

(1) $e_ie_{i+1} = e_i$, $i \geqq 0$.

(2) $e_1R \subset e_2R \subset \cdots$.

If we denote by I the R-submodule of $R[X]$ generated by $e_0, e_1X, e_2X^2, \cdots$, then I is a nonfinitely generated ideal of $R[X]$ such that $R[X]/I$ is R-projective.[2]

PROOF. I is an ideal of $R[X]$ for if $0 \leqq k$ is an integer and if

$$f = \sum_{i=0}^{n} r_i e_i X^i \in I,$$

then

$$X^k f = \sum_{i=0}^{n} r_i e_i X^{i+k} = \sum_{i=0}^{n} (r_i e_i) e_{i+k} X^{i+k} \in I.$$

Moreover, $R[X]/I \simeq \bigoplus_{i=0}^{\infty} (R/e_iR)$ is a projective R-module and if I were a finitely generated ideal of $R[X]$, then $c(I) = \bigcup_{i=0}^{\infty} e_iR$ would be a finitely generated ideal of R. Q.E.D.

We would like to close by posing the following problem: Determine the class $\mathscr{H}$ of rings R having the property that from $R[X]/I$ R-projective it follows that I is a finitely generated ideal. Since pure ideals of an $A(0)$-ring are finitely generated, it follows from Theorem 3 that each $A(0)$-ring belongs to $\mathscr{H}$. Moreover, one sees from Proposition 3 that if $R \in \mathscr{H}$, then R satisfies the a.c.c. on principal idempotent ideals. The following results shows that $A(0)$-rings satisfy the a.c.c. on principal idempotent ideals. This result is due to I. Sahaev **[9]**, but being unable to read Russian ourselves, we supply here our own easy proof.

PROPOSITION 4. *In order that R be an $A(0)$-ring it is necessary and sufficient that for each sequence $\{a_i\}_{i=0}^{\infty}$ of elements of R satisfying $a_ia_{i+1} = a_i$, $i \geqq 0$, the tower $a_0R \subseteq a_1R \subseteq \cdots$ terminates.*

PROOF. Suppose that such a sequence exists for which the tower $a_0R \subset a_1R \subset \cdots$ fails to terminate. Set $A = \bigcup_{i=0}^{\infty} a_iR$. Then by **[1**, Exercise 23, p. 65], A is a nonfinitely generated pure ideal of R and so R is not an $A(0)$-ring **[7**, Lemma 4.6].

Conversely, if R is not an $A(0)$-ring, there exists a pure ideal A of R which is not finitely generated. We will construct an infinite sequence of the required type. Let $a_0 = 0$ and suppose that $a_0, a_1, \cdots, a_n$ have been chosen so that $a_0R \subset a_1R \subset \cdots$ and $a_ia_{i+1} = a_i$, $0 \leqq i \leqq n-1$. Since A is pure, for each $a \in A$ there exists $b \in A$ with $ab = a$. Suppose that for all $b \in A$ with $ba_n = a_n$ we have that $bR = a_nR$. Then, if $a \in A$, by **[1**, Exercise 23, p. 65] there exists $c \in A$ such that $ca = a$ and $ca_n = a_n$. Thus, $a \in cR = a_nR$

[2] The idea for this proposition was given to the authors by a reviewer of an NSF proposal.

and so $A=a_nR$. It follows that there exists $a_{n+1} \in A$ with $a_na_{n+1}=a_n$ and $a_nR \subset a_{n+1}R$. Q.E.D.

It is not difficult to construct in $C([0, 1])$ a sequence $\{a_i\}_0^\infty$ such that $a_ia_{i+1}=a_i$ and $a_iR \subset a_{i+1}R$ for each i. Since $C([0, 1])$ is indecomposable, $C([0, 1])$ is not an $A(0)$-ring but does satisfy the a.c.c. on principal idempotent ideals. Thus, the class $\mathscr{H}$ falls somewhere between the class of $A(0)$-rings and the class of all rings satisfying the a.c.c. on principal idempotent ideals.

ADDED IN PROOF. The two problems specifically mentioned here have been solved and will appear in a subsequent issue of this journal.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KANSAS, LAWRENCE, KANSAS 66045