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AN EXTENSION OF HAYNSWORTH'S DETERMINANT INEQUALITY

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ABSTRACT. Let A and B be positive definite hermitian matrices of order n. This paper improves a lower bound given for $|A+B|$ by E. V. Haynsworth.

In [1] E. V. Haynsworth proves that if A and B are positive definite hermitian matrices of order n then

$$|A + B| \geqq \left(1 + \sum_{i=1}^{n-1} \frac{|B_i|}{|A_i|}\right) |A| + \left(1 + \sum_{i=1}^{n-1} \frac{|A_i|}{|B_i|}\right) |B|$$

where A_i, B_i denote the principal submatrices of order i in the upper left corner of A and B respectively. This note uses the same techniques developed in [1] to extend the inequality. This is done by applying the following elementary calculus result.

If $f(x)=ax+bx^{-1}$ with a and b positive, then $\min_{0<x<\infty} f(x)$ is achieved at $(b/a)^{1/2}$ with $f((b/a)^{1/2})=2(ab)^{1/2}$.

We also use the following lemma in the argument of the extended result.

LEMMA[1] [1, LEMMA 2, p. 514]. *If A and B are positive definite hermitian matrices of order n then*

$$|(A + B)/(A_{n-1} + B_{n-1})| \geqq |A|/|A_{n-1}| + |B|/|B_{n-1}|.$$

The extended inequality is as follows.

THEOREM. *Suppose A and B are positive definite hermitian matrices of order n, then*

$$|A + B| \geqq \left(1 + \sum_{i=1}^{n-1} \frac{|B_i|}{|A_i|}\right) |A| + \left(1 + \sum_{i=1}^{n-1} \frac{|A_i|}{|B_i|}\right) |B| + (2^n - 2n)(|A|\,|B|)^{1/2}.$$

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[1] We use the notation X/Y instead of XY^{-1} to be consistent with the source.

PROOF. The proof of the theorem is by induction on the order k of A and B. For $k=1$ the inequality reduces to $|A+B| \geqq |A|+|B|$ which is clearly true. Suppose therefore that the inequality holds for all pairs of positive definite hermitian matrices A and B of order k with $k<n$. Now suppose A and B are positive definite hermitian matrices of order n. Then, by applying the results of the lemma,

$$\begin{aligned} |A+B| &= |A_{n-1}+B_{n-1}|\,|(A+B)/(A_{n-1}+B_{n-1})| \\ &\geqq |A_{n-1}+B_{n-1}|\,[|A|/|A_{n-1}| + |B|/|B_{n-1}|] \end{aligned}$$

and hence, by the induction hypothesis,

$$\begin{aligned} |A+B| \geqq \Bigg[&\left(1+\sum_{i=1}^{n-2}\frac{|B_i|}{|A_i|}\right)|A_{n-1}| \\ &+\left(1+\sum_{i=1}^{n-2}\frac{|A_i|}{|B_i|}\right)|B_{n-1}| \\ &+[2^{n-1}-2(n-1)](|A_{n-1}|\,|B_{n-1}|)^{1/2}\Bigg]\left[\frac{|A|}{|A_{n-1}|}+\frac{|B|}{|B_{n-1}|}\right] \end{aligned}$$

$$\begin{aligned} = &\left(1+\sum_{i=1}^{n-2}\frac{|B_i|}{|A_i|}\right)|A| + \left(1+\sum_{i=1}^{n-2}\frac{|A_i|}{|B_i|}\right)|B| \\ &+\left(1+\sum_{i=1}^{n-2}\frac{|B_i|}{|A_i|}\right)\frac{|A_{n-1}|}{|B_{n-1}|}|B| + \left(1+\sum_{i=1}^{n-2}\frac{|A_i|}{|B_i|}\right)\frac{|B_{n-1}|}{|A_{n-1}|}|A| \\ &+[2^{n-1}-2(n-1)](|A_{n-1}|\,|B_{n-1}|)^{1/2}\left[\frac{|A|}{|A_{n-1}|}+\frac{|B|}{|B_{n-1}|}\right]. \end{aligned}$$

Thus by combining the first and part of the third expressions, the second and part of the fourth expressions we have

$$\begin{aligned} |A+B| \geqq &\left[1+\sum_{i=1}^{n-1}\frac{|B_i|}{|A_i|}\right]|A| + \left[1+\sum_{i=1}^{n-1}\frac{|A_i|}{|B_i|}\right]|B| \\ &+\sum_{i=1}^{n-2}\frac{|B_i|}{|A_i|}\frac{|A_{n-1}|\,|B|}{|B_{n-1}|} + \sum_{i=1}^{n-2}\frac{|A_i|}{|B_i|}\frac{|B_{n-1}|\,|A|}{|A_{n-1}|} \\ &+[2^{n-1}-2(n-1)](|A_{n-1}|\,|B_{n-1}|)^{1/2}\left[\frac{|A|}{|A_{n-1}|}+\frac{|B|}{|B_{n-1}|}\right]; \end{aligned}$$

hence by applying the calculus result termwise to a pair of corresponding terms in expressions three and four and also to expression five we have

$$|A+B| \geqq \left(1+\sum_{i=1}^{n-1}\frac{|B_i|}{|A_i|}\right)|A| + \left(1+\sum_{i=1}^{n-1}\frac{|A_i|}{|B_i|}\right)|B| + \sum_{i=1}^{n-2} 2(|A|\,|B|)^{1/2}$$
$$+ [2^{n-1} - 2(n-1)](|A_{n-1}|\,|B_{n-1}|)^{1/2}[2(|A|\,|B|/|A_{n-1}|\,|B_{n-1}|)^{1/2}]$$
$$\geqq \left(1+\sum_{i=1}^{n-1}\frac{|B_i|}{|A_i|}\right)|A| + \left(1+\sum_{i=1}^{n-1}\frac{|A_i|}{|B_i|}\right)|B| + (2^n - 2n)(|A|\,|B|)^{1/2}.$$

This is the conclusion of the Theorem.

As a direct corollary we have the following interesting inequality.

COROLLARY. *If A and B are positive definite hermitian matrices of order n then* $|A+B| \geqq |A|+|B|+(2^n-2)(|A|\,|B|)^{1/2}$.

PROOF. By Theorem 1,

$$|A+B| \geqq |A|+|B|+\left(\sum_{i=1}^{n-1}\frac{|B_i|}{|A_i|}\right)|A| + \left(\sum_{i=1}^{n-1}\frac{|A_i|}{|B_i|}\right)|B|$$
$$+ (2^n - 2n)(|A|\,|B|)^{1/2}$$

and so, by applying the calculus result on the third and fourth expressions,

$$|A+B| \geqq |A|+|B|+2(n-1)(|A|\,|B|)^{1/2} + (2^n-2n)(|A|\,|B|)^{1/2}$$
$$= |A|+|B|+(2^n-2)(|A|\,|B|)^{1/2}.$$

It is, of course, easily seen by continuity that the Corollary also holds for positive semidefinite hermitian matrices as well. Another feature of both results is that equality holds when $A=B$.

REFERENCE

1. E. V. Haynsworth, *Applications of an inequality for the Shur complement*, Proc. Amer. Math. Soc. **24** (1970), 512–516. MR **41** #241.

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