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THE MIN CONE OVER THE CIRCLE GROUP

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ABSTRACT. It is shown that the min cone over the circle group is the only inverse semigroup or Clifford semigroup on the 2-cell with an identity whose set of idempotents has a cut point.

A topological inverse semigroup S is a Hausdorff space together with a continuous associative multiplication in which each element has a unique inverse and whose idempotents commute. Such semigroups were studied in [5], which includes a characterization of inverse semigroups whose underlying space is an arc.

Considering inverse semigroups on the 2-cell, the purpose of this paper is to show that the min cone over the circle group is the only inverse semigroup with an identity on the 2-cell whose set of idempotents has a cut point. Unless otherwise stated, S will denote such a semigroup and E its set of idempotents. The closure of A will be denoted by A^* while other notations, definitions, and preliminary results can be found in [2], [3], and [5].

The existence of the kernel K in compact semigroups is well known, as are the facts that K is connected if S is connected, and that K consists of one $\mathscr{D}$-class. It follows that K is a single point and hence S has a zero.

LEMMA 1. *If e is a cut point of E, then $H(e)$ is a circle group.*

PROOF. Let r be the retraction of S onto E defined by $r(s)=ss^{-1}$ [5]. Thus $r^{-1}(e)=R(e)$, the $\mathscr{R}$-class of e, separates S and it follows that $\dim R(e)=\dim D(e)=1$ [3], [4]. Inversion restricted to $R(e)$ is a homeomorphism onto $L(e)$, so $\dim(L(e)\times R(e))=2$. The multiplication map m from $L(e)\times R(e)$ onto $D(e)$ lowers dimension by one, so it follows that there are elements $t\in L(e)$ and $s\in R(e)$ such that $m^{-1}(ts)\subset H(t)\times H(s)$ and $\dim m^{-1}(ts)\geqq 1$; hence $\dim H(t)=\dim H(s)=1$. By Green's translational lemmas, $H(e)$ is homeomorphic to $H(s)$, hence $\dim H(e)=1$ and $H(e)$ is a circle group [1].

Let T be a min thread from 1 to 0 [6]; clearly T contains e. By Exercise 7, p. 209 of [3], $SeS=eSe$ is a min cone over the circle group. In fact there

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is a maximal idempotent of T, say f, such that $SfS=fSf$ is a min cone over the circle group. Let e now denote this maximal idempotent. If $H(e)=\text{Bd } S$ then $e=1$ and we are through. Therefore suppose $e\neq 1$ and let $T_1=\{t\in T: e\leqq t\}$ and $C=\{s\in S: se=e\}$. Under this assumption, $E\backslash SeS\neq\varnothing$ and has no cut points, so let e_0 denote an interior point of $E\backslash SeS$.

LEMMA 2. *The center of S contains SeS.*

PROOF. For $x\in S\backslash SeS$ and $f\in E\cap SeS$, $f\leqq xx^{-1}$ and $xfx^{-1}\in E\cap H(f)$; but $H(f)$ is a group, so $xfx^{-1}=f$ and $xf=fx$. If $t\in SeS$, then $tx=(tt^{-1}t)x=t(t^{-1}t)x=(t^{-1}tx)t$ since $t^{-1}tx\in H(t)$ and SeS is commutative. Now $t^{-1}tx=xt^{-1}t$ since $t^{-1}t=tt^{-1}\in E\cap SeS$, so $tx=(t^{-1}tx)t=(xtt^{-1})t=xt$.

LEMMA 3. *The set C is a closed connected acyclic inverse subsemigroup of S.*

PROOF. For $c\in C$, $cT_1\subset C$ and is an arc from c to e. Therefore C is connected and it is easily shown that C is a closed connected inverse subsemigroup. The mapping H from $T_1\times C$ onto C defined by $H(t, c)=tc$ is continuous, where $H(1, c)=c$ and $H(e, c)=e$ for all $c\in C$. The generalized homotopy theorem [7] implies $H^n(C)\approx H^n(e)$ for all n, hence C is acyclic.

LEMMA 4. $1\notin(S\backslash C)^*$.

PROOF. If $\{t_n\}$ is a sequence in $S\backslash C$ converging to 1, then $\{t_ne_0\}$ converges to e_0 and there exists an m such that $t_me_0\in E\backslash SeS$. Hence $t_me_0\in C$; but $e=(t_me_0)e=t_m(e_0e)=t_me$, a contradiction, since $t_m\in S\backslash C$.

LEMMA 5. *If $x\in S\backslash C$ and $c\in C$, then xc, $cx\in S\backslash C$ and $xT\cap C=\varnothing$.*

PROOF. $ex=xe\neq e$ and $ce=ec=e$, so $(xc)e=x(ce)=xe\neq e$ and $(cx)e=e(cx)=ex\neq e$.

LEMMA 6. Bd *$S\cap C$ is an arc from a to a^{-1} for some $a\in C$.*

PROOF. Let $a\in\text{Bd } S\cap C$ such that $a\neq 1$. Since C is acyclic, Bd $S\not\subset C$, and since inversion is a homeomorphism on Bd S, it will suffice to show that one of the arcs of Bd S from a to 1 is contained in C. Let A and B be these arcs. Suppose there exist $x\in A\backslash C$ and $y\in B\backslash C$. Then the connected set $T_1\cup aT_1\subset C$ separates either x or y from 0, while $xT\cup yT$ is a connected subset of $S\backslash C$ containing x, y, and 0, a contradiction.

LEMMA 7. $\text{Bd}(S\backslash C)\cap C=aT_1\cup a^{-1}T_1$.

PROOF. Clearly $aT_1\cup a^{-1}T_1\subset\text{Bd}(S\backslash C)\cap C$, and since $1\notin aT_1\cup a^{-1}T_1$, this set separates S and the components separated from 0 are contained in C. Let $\{t_n\}$ be a sequence in $S\backslash C$ converging to a such that $t_n e \neq t_n^{-1}e$, and let H_n be the arc of $H(e)$ from $t_n e$ to $t_n^{-1}e$ containing e. Now consider the arcs $C_n = t_n T_1 \cup H_n \cup t_n^{-1}T_1$, noting $H_n \cap t_n T_1 = \{t_n e\}$, $H_n \cap t_n^{-1}T_1 = \{t_n^{-1}e\}$, and $t_n T_1 \cap t_n^{-1}T_1 = \varnothing$. The arcs C_n separate C from 0, and clearly $\{C_n\}$ converges to $aT_1 \cup a^{-1}T_1$. Now if $x \in S$ is such that x and 0 are not separated by $aT_1 \cup a^{-1}T_1$, then there is an m such that C_m separates x from C. Hence $x \notin C$ and $aT_1 \cup a^{-1}T_1 = \mathrm{Bd}(S\backslash C) \cap C$.

THEOREM 1. *S is the min cone over the circle group and $e=1$.*

PROOF. By Lemma 6, the Bd $S \cap C$ is an arc with end points a and a^{-1}. It follows from Lemmas 5 and 7 that $aa^{-1} \in aT_1 \cup a^{-1}T_1$. Therefore either $aa^{-1} = af$ for some $f \in T_1$ and $a^{-1} = a^{-1}aa^{-1} = a^{-1}af \in E$, a contradiction, or $aa^{-1} = a^{-1}f$ and $a = aa^{-1}a = a^{-1}fa \in E$, again a contradiction.

The referee observed that if S is a Clifford semigroup with identity on the 2-cell, whose set E of idempotents has a cut point, then Lemmas 1 and 2 follow with slight modifications, where e can again be considered the maximum idempotent of T for which eSe is a min cone over the circle group. Continuing the referee's suggestions, let $T_1 = \{t \in T : e \leqq t\}$. Then $\theta : T_1 \times S \to S$ defined by $\theta(t, x) = tx$ is a homotopy retract of S onto eSe and by the homology properties of the circle and disk, this implies $e \cdot \mathrm{Bd}\, S = H(e)$. Now if $g \in \mathrm{Bd}\, S$ is such that eg has infinite order, then multiplication by e maps the closed subgroup generated by g homomorphically onto $H(e)$, and since dimension cannot be raised by a homomorphism, $g \in H(e)$. Let $C = \{s \in S : se = e\}$; again Bd $S \cap C$ is a connected arc containing 1 with end points a and b. But there exist sequences of elements of infinite order in $H(e) \cap \mathrm{Bd}\, S$ converging to a and b. Hence $a = e = b$ and $e = 1$.

The following more general theorem now follows, although in the case of Clifford semigroups one must take into consideration the fact that idempotents need not commute.

THEOREM 2. *If S is an inverse semigroup or Clifford semigroup with identity on the 2-cell whose set of idempotents has a cut point, then S is the min cone over the circle group.*

The author has some results in the case when the set of idempotents has no cut point and in a later paper this case will be considered, with a view to completing a characterization of inverse and Clifford semigroups with identity on the 2-cell.

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REFERENCES

1. L. W. Anderson and R. P. Hunter, *The $\mathscr{H}$-equivalence in compact semigroups*, Bull. Soc. Math. Belg. **14** (1962), 274–296. MR **27** #1920.

2. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*. Vols. I, II, Math. Surveys, no. 7, Amer. Math. Soc., Providence, R.I., 1961, 1967. MR **24** #A2627; **36** #1558.

3. K. H. Hofmann and P. S. Mostert, *Elements of compact semigroups*, Charles E. Merrill, Columbus, Ohio, 1966. MR **35** #285.

4. W. Hurewicz and H. Wallman, *Dimension theory*, Princeton Math. Series, vol. 4, Princeton Univ. Press, Princeton, N.J., 1941. MR **3**, 312.

5. R. J. Koch and A. D. Wallace, *Notes on inverse semigroups*, Rev. Roumaine Math. Pures Appl. **9** (1964), 19–24. MR **31** #1662.

6. R. J. Koch, *Arcs in partially ordered spaces*, Pacific J. Math. **9** (1959), 723–728. MR **21** #7269.

7. A. D. Wallace, *Cohomology, dimension and mobs*, Summa Brasil. Math. **3** (1953), 43–55. MR **15**, 336.

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