BOUNDED APPROXIMATE UNITS AND BOUNDED APPROXIMATE IDENTITIES

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Abstract. In this paper we establish the equivalence of the general concept of bounded approximate units in a normed algebra with the traditionally used notion of a bounded approximate identity. Furthermore, we investigate pointwise-bounded approximate units in commutative normed algebras.

Let $A$ be a normed algebra. A net $\{e_\lambda\}_{\lambda \in \Lambda}$ of elements in $A$ is called a bounded left approximate identity in $A$ if there exists a constant $K$ such that $\|e_\lambda\| \leq K$ for all $\lambda \in \Lambda$ and $\lim_{\lambda \in \Lambda} e_\lambda x = x$ for all $x \in A$.

A normed algebra $A$ has bounded left approximate units if there exists a constant $K$ such that for every $x \in A$ and every $\varepsilon > 0$ there exists an element $u \in A$ (depending on $x$ and $\varepsilon$) such that $\|u\| \leq K$ and $\|x - ux\| < \varepsilon$.

We say that a normed algebra $A$ has pointwise-bounded left approximate units if for every $x \in A$ there exists a constant $K(x)$ such that for every $\varepsilon > 0$ there exists an element $u \in A$ (depending on $x$ and $\varepsilon$) such that $\|u\| \leq K(x)$ and $\|x - ux\| < \varepsilon$.

Obviously, every normed algebra with a bounded left approximate identity has bounded left approximate units. Theorem 1 states the converse; the proof given below was kindly communicated to the author by Sadahiro Saeki and replaces our original proof. The argument is a modification of that given by H. Reiter in [1, p. 30].

Furthermore, we show that a commutative normed algebra with pointwise-bounded approximate units has an approximate identity. The fact that we cannot assert the existence of a bounded approximate identity is illustrated by an example. But it turns out that the concept of pointwise-bounded approximate units is equivalent to the notion of a bounded approximate identity in commutative Banach algebras and also in commutative normed algebras which do not consist entirely of topological divisors of zero.

The general concept of approximate units in a normed algebra was...
considered (under various names) by several authors; e.g. H. Reiter [1, pp. 27ff.] and H. C. Wang [2].

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**Theorem 1.** A normed algebra $A$ has left approximate units bounded by a constant $K$ if and only if $A$ has a left approximate identity bounded by the same constant $K$.

**Proof.** Let $A$ be a normed algebra with left approximate units bounded by a constant $K$. With the usual convention about the formal role of 1, the assumption takes the following form: for every $x \in A$ and $\varepsilon > 0$ there exists an element $u \in A$ such that $\|u\| \leq K$ and $\|(1-u)x\| < \varepsilon$.

Now let $x_1, \cdots, x_n$ be any finite set of elements in $A$. Given $\varepsilon > 0$, we can choose successively $u_1, \cdots, u_n$ in $A$ such that

$$\|u_i\| \leq K \quad \text{and} \quad \|(1-u_1) \cdots (1-u_i)x_i\| < \varepsilon$$

for all $i=1, 2, \cdots, n$. Define $v$ in $A$ by $1-v = (1-u_n) \cdots (1-u_1)$. Then

$$\|x_i - vx_i\|$$

$$= \|[(1-u_n) \cdots (1-u_{i+1})] \cdot [(1-u_i) \cdots (1-u_1)x_i]\|$$

$$\leq (1 + K)^{n-i} \cdot \|(1-u_i) \cdots (1-u_1)x_i\| < (1 + K)^n \cdot \varepsilon.$$

Finally we choose $u$ in $A$ with $\|u\| \leq K$ and $\|v - uv\| < \varepsilon$. Then for each $i=1, 2, \cdots, n$ we have

$$\|x_i - ux_i\| \leq \|x_i - vx_i\| + \|(v - uv)x_i\| + \|u(x_i - vx_i)\|$$

$$\leq \|x_i - vx_i\| + \|v - uv\| \cdot \|x_i\| + K \cdot \|x_i - vx_i\|$$

$$< (1 + K)^n \cdot \varepsilon + \|x_i\| \cdot \varepsilon + (1 + K)^{n+1} \cdot \varepsilon.$$

Hence, for every finite set $x_1, \cdots, x_n$ of elements in $A$ and every $\varepsilon > 0$, there exists an element $u$ in $A$ such that $\|u\| \leq K$ and $\|x_i - ux_i\| < \varepsilon$ for $i=1, \cdots, n$. Using a well-known construction [1, p. 27] we conclude that the normed algebra $A$ has a left approximate identity bounded by the constant $K$.

**Theorem 2.** A commutative normed algebra with pointwise-bounded approximate units has an approximate identity (possibly unbounded).

A commutative normed algebra $A$ which does not consist entirely of topological divisors of zero has pointwise-bounded approximate units if and only if $A$ has a bounded approximate identity.

**Proof.** Let $A$ be a commutative normed algebra with pointwise-bounded approximate units. If $A$ does not consist entirely of topological
divisors of zero, let \( x_0 \) be an element in \( A \) which is not a topological divisor of zero; otherwise set \( x_0 = 0 \).

Now let \( x_1, \ldots, x_n \) be any finite set of elements in \( A \). Set

\[
K = K(x_0, x_1, \ldots, x_n) = \max\{K(x_0), K(x_1), \ldots, K(x_n)\}.
\]

Given \( \varepsilon > 0 \) there exist elements \( u_0, u_1, \ldots, u_n \) in \( A \) such that \( \|u_i\| \leq K \) and \( \|x_i - u_i x_i\| < \varepsilon \) for all \( i = 0, 1, \ldots, n \).

Define \( u \) in \( A \) by \( 1 - u = (1 - u_n) \cdots (1 - u_1)(1 - u_0) \). Then

\[
\|x_i - ux_i\| = \|(1 - u_n) \cdots (1 - u_1)(1 - u_0)x_i\| \leq (1 + K)^n \cdot \|(1 - u_i)x_i\| < (1 + K)^n \cdot \varepsilon.
\]

Hence, for every finite set \( x_1, \ldots, x_n \) of elements in \( A \) and every \( \varepsilon > 0 \) there exists an element \( u \) in \( A \) such that \( \|x_i - u x_i\| < \varepsilon \) for all \( i = 0, 1, \ldots, n \). If \( x_0 \) is not a topological divisor of zero, it follows from the inequality \( \|ux_0\| < \varepsilon + \|x_0\| \) that the elements \( u \) are bounded by some fixed constant. Thus the assertion of Theorem 2 follows.

The next example shows that we cannot in general assert the existence of a bounded approximate identity.

**Example.** Consider the commutative normed algebra

\[
A = \{(\lambda_1, \lambda_2, \cdots) \mid \lambda_i \text{ complex and } \lambda_i = 0 \text{ for almost all } i\}
\]

with coordinatewise algebraic operations and norm defined by

\[
\|(\lambda_1, \lambda_2, \cdots)\| = \max_i |i \cdot \lambda_i|.
\]

Then \( A \) has pointwise-bounded approximate units \( u = (1, \cdots, 1, 0, 0, \cdots) \) Obviously, \( A \) has no bounded approximate identity.

**Theorem 3.** A commutative Banach algebra \( A \) has pointwise-bounded approximate units if and only if \( A \) has a bounded approximate identity.

**Proof.** Let \( A \) be a commutative Banach algebra with pointwise-bounded approximate units. Define \( A_n = \{x \in A \mid \lim_i u_i x = x \text{ for some sequence } (u_i) \text{ in } A \text{ with } \|u_i\| \leq n\}, n = 1, 2, \cdots \).

\( A_n \) is a closed subset of \( A \). For if \( (x_i) \) is a sequence in \( A_n \) with \( \lim_j x_j = x \), then there exist sequences \( (u_{ij}) \) in \( A \) such that \( \|u_{ij}\| \leq n \) and \( \lim_i u_{ij} x_j = x_j \). Then

\[
\|x - u_{ij} x_j\| \leq \|x - x_j\| + \|x_j - u_{ij} x_j\| + \|u_{ij} x_j - u_{ij} x\| \leq \|x - x_j\| + \|x_j - u_{ij} x_j\| + \|u_{ij} \| \cdot \|x_j - x\| \leq (1 + n) \cdot \|x - x_j\| + \|x_j - u_{ij} x_j\|;
\]

choosing first \( j \) and then \( i \) large enough, it follows that \( \|x - u_{ij} x\| \) can be made arbitrarily small. Hence \( x \in A_n \).
Since $A$ is the union of the sets $A_n$, $n=1, 2, \cdots$, and $A$ is a Banach space, it follows from the Baire category theorem that some $A_n$ has nonempty interior. Thus $B(x_0, \delta) = \{x \in A | \|x-x_0\| < \delta\}$ is a subset of $A_m$ for some $x_0 \in A$, $\delta > 0$ and $m$. We will show that $B(0, \delta) = \{x \in A | \|x\| < \delta\}$ is a subset of $A_{(2+m)m}$. Let $x \in B(0, \delta)$; then $x = (x+x_0)-x_0$ with $x+x_0$ and $x_0$ in $B(x_0, \delta)$. Hence there exist sequences $(u_i)$ and $(v_i)$ in $A$ such that $\|u_i\| \leq m$, $\|v_i\| \leq m$, $\lim_i u_i(x+x_0) = x+x_0$ and $\lim_i v_i x_0 = x_0$. Set $w_i = u_i + v_i - u_i v_i$; then $(w_i)$ is a sequence in $A$ with $\|w_i\| \leq (2+m)m$ and $\lim_i w_i x = x$; i.e. $x$ is in $A_{(2+m)m}$.

Since $\lambda \cdot A_{(2+m)m}$ is a subset of $A_{(2+m)m}$ for any scalar $\lambda$, it follows that $A_{(2+m)m} = A$; i.e. $A$ has bounded approximate units and so, by Theorem 1, $A$ has a bounded approximate identity.

Added in proof. It was stated by M. Altman (Contracteurs dans les algèbres de Banach, C.R. Acad. Sci. Paris Sér. A 274 (1972), A399–A400; Lemme 1) that every Banach algebra with bounded left approximate units has a bounded left approximate identity. The proof will appear in his paper on Contractors, approximate identities and factorization in Banach algebras in the Pacific J. Math.

It was proved by Teng-sun Liu, Arnoud van Rooij and Ju-kwei Wang (Projections and approximate identities for ideals in group algebras, Trans. Amer. Math. Soc. 175 (1973), 469–482; Lemma 12) that every commutative Banach algebra with pointwise-bounded approximate units has bounded approximate units.

REFERENCES


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