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BOUNDED APPROXIMATE UNITS AND BOUNDED APPROXIMATE IDENTITIES

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ABSTRACT. In this paper we establish the equivalence of the general concept of bounded approximate units in a normed algebra with the traditionally used notion of a bounded approximate identity. Furthermore, we investigate pointwise-bounded approximate units in commutative normed algebras.

Let A be a normed algebra. A net $\{e_\lambda\}_{\lambda\in\Lambda}$ of elements in A is called a *bounded left approximate identity in A* if there exists a constant K such that $\|e_\lambda\| \leqq K$ for all $\lambda \in \Lambda$ and $\lim_{\lambda\in\Lambda} e_\lambda x = x$ for all $x \in A$.

A normed algebra A has *bounded left approximate units* if there exists a constant K such that for every $x \in A$ and every $\varepsilon > 0$ there exists an element $u \in A$ (depending on x and ε) such that $\|u\| \leqq K$ and $\|x-ux\| < \varepsilon$.

We say that a normed algebra A has *pointwise-bounded left approximate units* if for every $x \in A$ there exists a constant $K(x)$ such that for every $\varepsilon > 0$ there exists an element $u \in A$ (depending on x and ε) such that $\|u\| \leqq K(x)$ and $\|x-ux\| < \varepsilon$.

Obviously, every normed algebra with a bounded left approximate identity has bounded left approximate units. Theorem 1 states the converse; the proof given below was kindly communicated to the author by Sadahiro Saeki and replaces our original proof. The argument is a modification of that given by H. Reiter in [**1**, p. 30].

Furthermore, we show that a commutative normed algebra with pointwise-bounded approximate units has an approximate identity. The fact that we cannot assert the existence of a *bounded* approximate identity is illustrated by an example. But it turns out that the concept of pointwise-bounded approximate units is equivalent to the notion of a bounded approximate identity in commutative Banach algebras and also in commutative normed algebras which do not consist entirely of topological divisors of zero.

The general concept of approximate units in a normed algebra was

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considered (under various names) by several authors; e.g. H. Reiter [**1**, pp. 27ff.] and H. C. Wang [**2**].

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THEOREM 1. *A normed algebra A has left approximate units bounded by a constant K if and only if A has a left approximate identity bounded by the same constant K.*

PROOF. Let A be a normed algebra with left approximate units bounded by a constant K. With the usual convention about the formal role of 1, the assumption takes the following form: for every $x \in A$ and $\varepsilon > 0$ there exists an element $u \in A$ such that $\|u\| \leqq K$ and $\|(1-u)x\| < \varepsilon$.

Now let $x_1, \cdots, x_n$ be any finite set of elements in A. Given $\varepsilon > 0$, we can choose *successively* $u_1, \cdots, u_n$ in A such that

$$\|u_i\| \leqq K \quad \text{and} \quad \|(1 - u_i) \cdot \cdots \cdot (1 - u_2) \cdot (1 - u_1)x_i\| < \varepsilon$$

for all $i = 1, 2, \cdots, n$. Define v in A by $1 - v = (1 - u_n) \cdot \cdots \cdot (1 - u_2) \cdot (1 - u_1)$. Then

$$\begin{aligned}
&\|x_i - vx_i\| \\
&\qquad = \|[(1 - u_n) \cdot \cdots \cdot (1 - u_{i+1})] \cdot [(1 - u_i) \cdot \cdots \cdot (1 - u_1)x_i]\| \\
&\qquad \leqq (1 + K)^{n-i} \cdot \|(1 - u_i) \cdot \cdots \cdot (1 - u_1)x_i\| < (1 + K)^n \cdot \varepsilon.
\end{aligned}$$

Finally we choose u in A with $\|u\| \leqq K$ and $\|v - uv\| < \varepsilon$. Then for each $i = 1, 2, \cdots, n$ we have

$$\begin{aligned}
\|x_i - ux_i\| &\leqq \|x_i - vx_i\| + \|(v - uv)x_i\| + \|u(x_i - vx_i)\| \\
&\leqq \|x_i - vx_i\| + \|v - uv\| \cdot \|x_i\| + K \cdot \|x_i - vx_i\| \\
&< (1 + K)^n \cdot \varepsilon + \|x_i\| \cdot \varepsilon + (1 + K)^{n+1} \cdot \varepsilon.
\end{aligned}$$

Hence, for every finite set $x_1, \cdots, x_n$ of elements in A and every $\varepsilon > 0$, there exists an element u in A such that $\|u\| \leqq K$ and $\|x_i - ux_i\| < \varepsilon$ for $i = 1, \cdots, n$. Using a well-known construction [**1**, p. 27] we conclude that the normed algebra A has a left approximate identity bounded by the constant K.

THEOREM 2. *A commutative normed algebra with pointwise-bounded approximate units has an approximate identity (possibly unbounded).*

A commutative normed algebra A which does not consist entirely of topological divisors of zero has pointwise-bounded approximate units if and only if A has a bounded approximate identity.

PROOF. Let A be a commutative normed algebra with pointwise-bounded approximate units. If A does not consist entirely of topological divisors of zero, let x_0 be an element in A which is not a topological divisor of zero; otherwise set $x_0=0$.

Now let $x_1, \cdots, x_n$ be any finite set of elements in A. Set

$$K = K(x_0, x_1, \cdots, x_n) = \max\{K(x_0), K(x_1), \cdots, K(x_n)\}.$$

Given $\varepsilon>0$ there exist elements $u_0, u_1, \cdots, u_n$ in A such that $\|u_i\| \leqq K$ and $\|x_i - u_i x_i\| < \varepsilon$ for all $i=0, 1, \cdots, n$.

Define u in A by $1-u=(1-u_n) \cdot \cdots \cdot (1-u_1)(1-u_0)$. Then

$$\begin{aligned}\|x_i - ux_i\| &= \|(1-u_n) \cdot \cdots \cdot (1-u_1)(1-u_0)x_i\| \\ &\leqq (1+K)^n \cdot \|(1-u_i)x_i\| < (1+K)^n \cdot \varepsilon.\end{aligned}$$

Hence, for every finite set $x_1, \cdots, x_n$ of elements in A and every $\varepsilon>0$ there exists an element u in A such that $\|x_i - ux_i\| < \varepsilon$ for all $i=0, 1, \cdots, n$. If x_0 is not a topological divisor of zero, it follows from the inequality $\|ux_0\| < \varepsilon + \|x_0\|$ that the elements u are bounded by some fixed constant. Thus the assertion of Theorem 2 follows.

The next example shows that we cannot in general assert the existence of a *bounded* approximate identity.

EXAMPLE. Consider the commutative normed algebra

$$A = \{(\lambda_1, \lambda_2, \cdots) \mid \lambda_i \text{ complex and } \lambda_i = 0 \text{ for almost all } i\}$$

with coordinatewise algebraic operations and norm defined by

$$\|(\lambda_1, \lambda_2, \cdots)\| = \max_i |i \cdot \lambda_i|.$$

Then A has pointwise-bounded approximate units $u_i = (1, \cdots, 1, 0, 0, \cdots)$ Obviously, A has no *bounded* approximate identity.

THEOREM 3. *A commutative Banach algebra A has pointwise-bounded approximate units if and only if A has a bounded approximate identity.*

PROOF. Let A be a commutative Banach algebra with pointwise-bounded approximate units. Define $A_n = \{x \in A \mid \lim_i u_i x = x$ for some sequence (u_i) in A with $\|u_i\| \leqq n\}$, $n=1, 2, \cdots$.

A_n is a closed subset of A. For if (x_j) is a sequence in A_n with $\lim_j x_j = x$, then there exist sequences $(u_{ij})_i$ in A such that $\|u_{ij}\| \leqq n$ and $\lim_i u_{ij} x_j = x_j$. Then

$$\begin{aligned}\|x - u_{ij}x\| &\leqq \|x - x_j\| + \|x_j - u_{ij}x_j\| + \|u_{ij}x_j - u_{ij}x\| \\ &\leqq \|x - x_j\| + \|x_j - u_{ij}x_j\| + \|u_{ij}\| \cdot \|x_j - x\| \\ &\leqq (1+n) \cdot \|x - x_j\| + \|x_j - u_{ij}x_j\|;\end{aligned}$$

choosing first j and then i large enough, it follows that $\|x - u_{ij}x\|$ can be made arbitrarily small. Hence $x \in A_n$.

Since A is the union of the sets A_n, $n=1, 2, \cdots$, and A is a Banach space, it follows from the Baire category theorem that some A_n has nonempty interior. Thus $B(x_0, \delta)=\{x \in A \mid \|x-x_0\|<\delta\}$ is a subset of A_m for some $x_0 \in A$, $\delta>0$ and m. We will show that $B(0, \delta)=\{x \in A \mid \|x\|<\delta\}$ is a subset of $A_{(2+m)m}$. Let $x \in B(0, \delta)$; then $x=(x+x_0)-x_0$ with $x+x_0$ and x_0 in $B(x_0, \delta)$. Hence there exist sequences (u_i) and (v_i) in A such that $\|u_i\| \leqq m$, $\|v_i\| \leqq m$, $\lim_i u_i(x+x_0)=x+x_0$ and $\lim_i v_i x_0=x_0$. Set $w_i=u_i+v_i-u_i v_i$; then (w_i) is a sequence in A with $\|w_i\| \leqq (2+m)m$ and $\lim_i w_i x=x$; i.e. x is in $A_{(2+m)m}$.

Since $\lambda \cdot A_{(2+m)m}$ is a subset of $A_{(2+m)m}$ for any scalar λ, it follows that $A_{(2+m)m}=A$; i.e. A has bounded approximate units and so, by Theorem 1, A has a bounded approximate identity.

ADDED IN PROOF. It was stated by M. Altman (*Contracteurs dans les algèbres de Banach*, C.R. Acad. Sci. Paris Sér. A **274** (1972), A399–A400; Lemme 1) that every Banach algebra with bounded left approximate units has a bounded left approximate identity. The proof will appear in his paper on *Contractors, approximate identities and factorization in Banach algebras* in the Pacific J. Math.

It was proved by Teng-sun Liu, Arnoud van Rooij and Ju-kwei Wang (*Projections and approximate identities for ideals in group algebras*, Trans. Amer. Math. Soc. **175** (1973), 469–482; Lemma 12) that every commutative Banach algebra with pointwise-bounded approximate units has bounded approximate units.

REFERENCES

1. H. Reiter, *L^1-algebras and Segal algebras*, Springer-Verlag, Berlin and New York, 1971.

2. H. C. Wang, *Nonfactorization in group algebras*, Studia Math. **42** (1972), 231–241.

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