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COMPACT 2-MANIFOLDS AS MAXIMAL IDEAL SPACES

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ABSTRACT. It is shown that every compact 2-manifold is (homeomorphic to) the maximal ideal space of an antisymmetric algebra which is Dirichlet on its Šilov boundary.

Introduction. Wermer and Browder [3] have shown that the sphere and real projective plane can be obtained as the maximal ideal space of an antisymmetric Dirichlet algebra. The conjecture then arises that every compact 2-manifold can be so realized. We prove this conjecture. We follow, in part, the formats of [1] and [3].

Main result. As is well known, every compact 2-manifold may be realized from the unit disk by making appropriate identifications of subarcs of the unit circle. Given a manifold M, let us choose all of our identification homeomorphisms to be singular, i.e. each takes a set of Lebesgue measure 0 into a set of full measure. Let us call Ψ the union of these homeomorphisms, and C, which is a subset of the unit circle Γ, the domain of Ψ.

Recall a function algebra A on a space X is *Dirichlet* if the real parts of the function in A are dense in the real continuous functions on X.

DEFINITION 1. Let $B=\{f\in C(\Delta):f$ is analytic in $\mathrm{int}(\Delta)\}$, where Δ=closed unit disk.

DEFINITION 2. Let $A=\{f\in B:f(z)=f(\psi(z))$ for all $z\in C\}$.

DEFINITION 3. $A_\psi=A$ restricted to C.

THEOREM. *A is a Dirichlet algebra on C; its maximal ideal space is M and its Šilov boundary is C.*

DEFINITION 4. A (complex Borel) measure ν on Γ is odd if, for each Borel set $E\subset C$, $\nu(E)=-\nu(\Psi(E))$.

DEFINITION 5. H denotes the class of measures of the form $g(z)\,dz$ where g is any function in the L^1 closure of B restricted to Γ.

DEFINITION 6. W is the space of measures $\mu+\nu$ with $\mu\in H$, ν odd; $\overline{W}$ is the weak* closure of W in the space of measures on Γ.

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DEFINITION 7. A measure λ on Γ annihilates A if $\int f\,d\lambda=0$, all $f \in A$. Clearly, every measure in W annihilates A. Also, if λ annihilates A, then $\lambda \in \overline{W}$.

LEMMA 1. *If $\mu \in H$, ν is odd, then $\|\nu\| \leqq 16\|\mu+\nu\|$.*

PROOF. Let E be any Borel subset of C and let m represent 1-dimensional Lebesgue measure. Then there are disjoint sets F and G with $E=F\cup G$, $m(F)=m(\psi(G))=0$. Let $K=\|\mu+\nu\|$, then $|\nu(F)|=|\nu(F)+\mu(F)|\leqq K$, since μ is absolutely continuous. $|\nu(G)|=|\nu(\psi)(G)|=|\mu+\nu(\psi(G))|\leqq K$, for the same reason. Hence $\|\nu\|\leqq 16K$. Q.E.D.

LEMMA 2. *Then $W=\overline{W}$.*

PROOF. $Q=[\mu+\nu:\mu \in H,\ \nu \text{ odd},\ \|\mu\|\leqq 1,\ \|\nu\|\leqq 1]$ is compact. The Kreĭn-Smulian theorem [4, p. 429] then implies that $W=\overline{W}$. Q.E.D.

LEMMA 3. *If ν is an odd measure, then ν is absolutely continuous with respect to arc length iff $\nu=0$.*

PROOF. Suppose ν is absolutely continuous. Let E be a Borel subset of C, then there are disjoint subsets F and G with $E=F\cup G$. Hence $\nu(F)=0$, since ν is absolutely continuous. $\nu(G)=-\nu(\Psi(G))=0$ for the same reason. Hence $\nu(E)=0$. Q.E.D.

LEMMA 4. *Every real annihilator λ of A has the form $\lambda=\nu$.*

PROOF. Since W is weak* closed, we conclude that if λ is a measure on Γ which annihilates A, then $\lambda=\mu+\nu$, $\mu \in H$, ν odd. Write $\mu=\mu_1+i\mu_2$, $\nu=\nu_1+i\nu_2$, with $\mu_1, \mu_2, \nu_1, \nu_2$ real. If λ is real, $\mu_2+\nu_2=0$. Hence ν_2 is absolutely continuous, hence 0. Then $\mu=0$, while $\nu=\nu_1$.

LEMMA 5. *A separates the points of C. Further given z_1, z_2 with $z_1 \in \Delta$, $z_2 \in \Delta - C$ and $z_1 \neq z_2$, then there exists an f in A such that $f(z_1)\neq f(z_2)$.*

PROOF. Let τ_1, τ_2 be two points of C and let $\delta\tau_1$, $\delta\tau_2$ be the point masses at τ_1, τ_2, respectively. Unless A separates τ_1, τ_2, $\delta\tau_1-\delta\tau_2$ will be a real annihilating measure which is not in W. Now suppose z_1, z_2 are interior to Δ and A fails to separate them. Let σ_1, σ_2 be the harmonic measure for z_1, z_2 respectively. Then $\sigma_1-\sigma_2$ would be a real annihilating measure. Hence $\sigma_1-\sigma_2=\nu$, ν odd. However $\sigma_1-\sigma_2$ is absolutely continuous. A contradiction. Finally, if $z_1 \in C$ and z_2 is interior to Δ a similar argument applies. Q.E.D.

LEMMA 6. *The space of maximal ideals of $A_\psi(A)$ is homeomorphic to M.*

PROOF. It must be shown that, if h is a homomorphism of A onto the complex numbers, then h is evaluation at some point of M. If h is not evaluation at any point of M, then for each $z \in \Delta$, there is an f_z with $h(f_z)=0$, $f_z(z)\neq 0$. Since M is compact, we can select a finite number of functions f_1, f_n in A such that $h(f_i)$ and open sets Δ_i in Δ such that $\bigcup_1^n \Delta_i=\Delta$ and $f_i\neq 0$ in the closure $\bar{\Delta}_i$ of Δ_i. Let σ be a representing measure for h on Γ, i.e. $h(f)=\int f\,d\sigma$, all $f\in A$. Then $\int f\cdot f_i\,d\sigma= h(f\cdot f_i)=h(f)\cdot h(f_i)=0$ $i=1, 2, \cdots, n$, $f \in A$. Thus $f_i\,d\sigma$ annihilates A, therefore $f_i\,d\sigma=d\mu_i+d\nu_i$, $\mu_i \in H$, ν_i odd. Hence, $f_j(d\mu_i+d\nu_i)=f_jf_i\,d\sigma= f_i(d\mu_j+d\nu_j)$ and so $f_j\,d\mu_i-f_i\,d\mu_j=f_i\,d\nu_j-f_j\,d\nu_i$. Since the right side is odd and the left side is absolutely continuous, both sides vanish. Let $\Phi_i\,dz$ denote the measure in H such that $d\mu_i=\Phi_i\,dz$. Then $f_j\Phi_i=f_i\Phi_j$ a.e. on Γ and so $f_j\Phi_i=f_i\Phi_j$ also for $z \in \operatorname{int}\Delta$. We can therefore unambiguously define Φ on Δ by $\Phi(z)=\Phi_i(z)(f_i(z))^{-1}$ for $z\in\Delta_i$. Then $\Phi\,dz \in H$.

We define a measure ν on Γ by $d\nu=(f_i)^{-1}\,d\nu_i$ on Γ by $d\nu=(f_i)^{-1}\,d\nu_i$ on $\Gamma\cap\Delta_i$. Then ν is well defined and odd. Then $f_i\,d\sigma=f_i\Phi\,dz+f_i\,d\nu$ on $\Gamma\cap\Delta_i$.

Since $f_i\neq 0$ on Δ_i we deduce $d\sigma=\Phi\,dz+d\nu$. But then $1=\int d\sigma=\int\Phi\,dz+ \int d\nu=0$. Contradiction. Q.E.D.

PROOF OF THEOREM. Combine Lemmas 4, 5, and 6. Q.E.D.

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