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EQUATIONAL THEORIES OF ALGEBRAS WITH DISTRIBUTIVE CONGRUENCES

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ABSTRACT. If an equational class of algebras has the distributive or permutable congruence property then it is well known that it satisfies certain conditions, known as Mal′cev-type conditions. In this note such Mal′cev-type conditions are used to find minimal bases for certain equational theories of algebras. A typical result states that every finitely based equational theory of algebras with distributive and permutable congruences is one-based.

1. Let us start by recalling a result of B. Jónsson **[2]** that an equational class $\mathscr{K}$ of algebras has the distributive congruence property if and only if for some natural number $n \geqq 2$ $\mathscr{K}$ admits $n+1$ ternary polynomials $t_0, t_1, \cdots, t_n$ satisfying the identities

$$\Delta_n: \begin{array}{ll} t_k(x, y, x) = x & (k = 0, 1, \cdots, n), \\ t_k(x, x, z) = t_{k+1}(x, x, z) & \text{if } k \text{ is even}, \\ t_k(x, z, z) = t_{k+1}(x, z, z) & \text{if } k \text{ is odd}, \\ t_0(x, y, z) = x, \qquad t_n(x, y, z) = z. & \end{array}$$

For brevity, the conditions Δ_n are usually called "n-distributivity". The following result was first obtained by K. Baker (Theorem 5.1 in **[1]**). We insert a quite elementary proof here.

LEMMA 1 (K. BAKER). *Relative to Δ_n, any finite set of identities can be equivalently expressed by a single identity.*

PROOF. As Baker has noted, in the presence of Δ_n, any identity "$f=g$" can be equivalently expressed by the $n-1$ identities

$$t_k(u, f, v) = t_k(u, g, v), \qquad k = 1, \cdots, n-1.$$

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Now, let $f_i(x_{i_1}, \cdots, x_{i_n}) = g_i(x_{i_1}, \cdots, x_{i_n})$, $i = 1, \cdots, m$, be any set of m identities.

Let $\Sigma = \{t_k(u_k, f_i, v_k) = t_k(u_k, g_i, v_k) \mid 1 \leqq i \leqq m, 1 \leqq k \leqq n-1\}$. Abbreviate $t_k(u_k, f_i, v_k)$ by a_{i_k} and $t_k(u_k, g_i, v_k)$ by b_{i_k}. Relative to Δ_n, Σ has two properties: (i) Σ holds iff the identities $f_i = g_i$ hold for all $i = 1, \cdots, m$; (ii) each polynomial a_{i_k}, b_{i_k} is idempotent (i.e.)

$$a_{i_k}(x, \cdots, x) = b_{i_k}(x, \cdots, x) = x.$$

Hence, Σ can be equivalently expressed by a single identity, relative to Δ_n. Let us illustrate this with two identities (which is sufficient): let

$$a_1(x_1, \cdots, x_n), \quad b_1(x_1, \cdots, x_n), \quad a_2(y_1, \cdots, y_m), \quad b_2(y_1, \cdots, y_m)$$

be polynomials such that

$$\Delta_n \models a_1(x, \cdots, x) = b_1(x, \cdots, x) = a_2(x, \cdots, x) = b_2(x, \cdots, x) = x.$$

Then the set of two identities $a_1(x_1, \cdots, x_n) = b_1(x_1, \cdots, x_n)$ and $a_2(y_1, \cdots, y_m) = b_2(y_1, \cdots, y_m)$ is clearly equivalent to the identity

$$\begin{aligned} &a_1(a_2(y_{11}, \cdots, y_{1m}), a_2(y_{21}, \cdots, y_{2m}), \cdots, a_2(y_{n1}, \cdots, y_{nm})) \\ &\qquad = b_1(b_2(y_{11}, \cdots, y_{1m}), b_2(y_{21}, \cdots, y_{2m}), \cdots, b_2(y_{n1}, \cdots, y_{nm})). \end{aligned}$$

This completes the proof of the lemma.

LEMMA 2. *If $n > 2$ then the set of identities Δ_n is equivalent to a set of $n-1$ identities.*

PROOF. The language of Δ_n consists of $n-1$ ternary operations $t_1, t_2, \cdots, t_{n-1}$ (t_0 and t_n were introduced so that the expressions Δ_n become elegant) and each t_i is idempotent. Hence the theory of Δ_n is an equational theory of idempotent algebras of type $\langle 3, 3, \cdots, 3 \rangle$ ($n-1$ times) and hence, by [**6**], is $n-1$ based.

From Lemmas 1 and 2, we get the

COROLLARY 1. *If $n > 2$ then every finitely based equational class of algebras with the n-distributivity property is n-based.*

When $n = 2$, Δ_n consists of just one ternary polynomial $t(x, y, z)$ satisfying the identities

$$t(x, y, x) = t(x, x, z) = t(z, x, x) = x;$$

i.e. "t" is a majority polynomial.

LEMMA 3. *A ternary polynomial symbol "t" is a majority polynomial iff it satisfies the identity*

$$t(t(x, y, y), u, t(t(x, y, y), y, z)) = y. \tag{1}$$

PROOF. The "only if" part being obvious we need prove only the "if" part.

Putting $z=t(t(x, y, y), y, w)$ in (1) we get

$$t(t(x, y, y), y, z) = t(t(x, y, y), y, t(t(x, y, y), y, w)) = y$$

by (1) and hence (1) reduces to

$$t(t(x, y, y), u, y) = y. \tag{2}$$

Putting $x=t(a, y, y)$ in (2) we notice that the term $t(x, y, y)$ becomes $t(t(a, y, y), y, y)$ which is simply "y" by (2) and hence we get

$$t(y, u, y) = y. \tag{3}$$

Now put $z=t(x, y, y)$ in (1). By two successive applications of (3) we have

$$t(x, y, y) = y \tag{4}$$

and hence (1) reduces to $t(y, u, t(y, y, z))=y$. Finally the substitution $u=t(y, y, z)$ and an application of (4) yields the last of the majority conditions $t(y, y, z)=y$, and this completes the proof of the lemma.

Combining this result with Lemma 1 along with Corollary 1 we get the following

THEOREM 1. *A finitely based equational class of algebras with the n-distributive congruence property is n-based.*

REMARK. If $n=2$ the above result is the best possible, since, for example, the theory of distributive lattices is not one-based (see [**5**]). However, it is not known whether the above is the best possible in other cases.

2. If an equational class $\mathscr{K}$ of algebras has both the distributive congruence property and the permutable congruence property then A. F. Pixley ([**7**], [**8**]) has shown that $\mathscr{K}$ admits a ternary polynomial p satisfying

$$p(y, y, x) = p(x, z, z) = p(x, u, x) = x \tag{5}$$

and conversely.

Let $\mathscr{K}$ be an equational class of algebras with distributive and permutable congruences. Of course, by Lemma 1, it follows that, in the presence of the identities (5), the validity of any finite number of identities is equivalent to that of a single one. However, the identities (5) are so nice that this becomes an easy verification. Indeed, in presence of (5), any identity $f(x_1, \cdots, x_n)=g(x_1, \cdots, x_n)$ is equivalent to the identity

$p(u, f, g)=u$. Now, let

$$(6) \qquad f(y_1, \cdots, y_n) = y_1, \qquad g(z_1, \cdots, z_m) = z_1,$$

be two identities. Consider the identity

$$(7) \qquad p(x, f, y_1) = p(x, g, z_1).$$

Clearly (6) implies (7). Conversely, identifying $x=y_1=g(z_1, \cdots, z_m)$ in (7) and using the first and the last of the identities (5) we get $g(z_1, \cdots, z_m)=z_1$ and similarly we have $f(y_1, \cdots, y_n)=y_1$.

THEOREM 2. *Every finitely based equational theory of algebras with distributive and permutable congruences is one-based.*

PROOF. Thus it is sufficient if we show that these four identities (5) and $f(z, z_1, \cdots, z_n)=z$ can be equivalently expressed by a single identity. Now consider

$$(8) \qquad p(p(u, u, x), p(f, y, z), z) = x.$$

Certainly the four identities with which we started jointly imply (8). Thus it remains to show that (8) implies the four identities.

Define $f_1 \equiv f(x, z_1, \cdots, z_n)$ and put $u=p(f_1, f_1, x)$ in (8). Then

$$p(u, u, x) = p(p(f_1, f_1, x), p(f_1, f_1, x), x) = x$$

by (8) and hence (8) reduces to

$$(9) \qquad p(x, p(f, y, z), z) = x.$$

Thus, by (8) and (9) we have

$$(10) \qquad p(u, u, x) = x.$$

Now the substitution $x=p(f, y, z)$ in (9) along with one application of (10) yields

$$(11) \qquad p(f, y, z) = z.$$

By (9) and (11) we have

$$(12) \qquad p(x, z, z) = x.$$

The substitution $y=z$ in (11) along with (12) yields the identity

$$f(z, z_1, \cdots, z_n) = z$$

and finally by (11) and the above we have $p(z, y, z)=z$ and this completes the proof of the theorem.

Baker has shown that every finite algebra of an arbitrary equational class with distributive congruences has a finite base (see also M. Makkai, Notices Amer. Math. Soc. **20** (1972), p. A-254). This, coupled with our Theorem 2, yields the following

COROLLARY 2. *Every finite algebra of an equational class with distributive and permutable congruences is one-based.*

For example, all quasi-primal algebras are such (see e.g. [**8**]) and hence are one-based. For primal algebras, this result was previously obtained by G. Grätzer and R. McKenzie [**3**]. As stated before, Corollary 2 is not true once we drop the permutable congruence property from the hypothesis. On the other hand, T. C. Green [**4**] has shown that, given n, there are finitely based equational theories with permutable congruences (in fact, definitionally equivalent to groups) which are n-based but not $n-1$ based. This shows that the distributive congruence property cannot be dropped either.

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