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GROUPS WITH NORMAL SUBGROUPS POSSESSING SUBNORMAL COMPLEMENTS

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ABSTRACT. J. Wiegold has characterized groups in which every normal subgroup is a direct factor as the restricted direct products of simple groups. In this paper, it is proved that for a group G to have the structure above, it is sufficient that every normal subgroup of G has a subnormal complement in G.

1. **Introduction.** A subgroup A of a group G is said to be complemented in G if there is a subgroup B of G such that $G=AB$ and $A\cap B=1$. The subgroup B is called a complement of A in G. Adopting a notation of C. Christensen [**2**], we shall call G an nD-group (nS-group, nC-group) if every normal subgroup of G has a normal complement (a subnormal complement, some complement). It is clear that

$$[nD] \subseteq [nS] \subseteq [nC]$$

where $[X]$ denotes the class of X-groups. J. Wiegold [**5**] has characterized nD-groups as the (restricted) direct products of simple groups. The much wider class of nC-groups has been studied by C. Christensen ([**2**], [**3**]), S. N. Černikov [**1**], N. T. Dinerstein [**4**] and D. I. Zaicev [**6**]. Our purpose is to show that the classes $[nD]$ and $[nS]$ are identical, i.e. we shall prove the following

THEOREM. *Every nS-group is an nD-group.*

The proof employs Wiegold's result cited above and a theorem of Černikov [**1**, Theorem 13, Corollary 2] which states that a locally nilpotent nC-group is a (restricted) direct product of cyclic groups of prime orders.

2. **Lemmas.** In all that follows, by the direct product of a set of groups we shall mean the restricted direct product. The symbol $N\lhd G$ will denote

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the fact that N is a normal subgroup of G. For any two elements a and b of G, $[a, b]$ will denote the element $a^{-1}b^{-1}ab$.

LEMMA 1. *Let G be an nS-group, $A \lhd G$ and B a complement of A in G. Then B is an nS-group.*

PROOF. Since $G=AB$ and $A\cap B=1$, $B\simeq G/A$. Let N be a normal subgroup of G containing A and K a subnormal complement of N. Then in the factor group G/A, KA/A is a subnormal complement of N/A. Thus, G/A, and hence B, is an nS-group.

LEMMA 2. *Let G be a group, K a subnormal subgroup of G and M a nonabelian minimal normal subgroup of K. Then M^G (the normal closure of M in G) is a minimal normal subgroup of G.*

3. **Proof of the theorem.** Let G be an nS-group. Denote by A the subgroup generated by the nonabelian minimal normal subgroups of G and by B that generated by the abelian subnormal subgroups of G. Then $A \lhd G$, and since conjugates of subnormal subgroups are subnormal, we also have $B \lhd G$.

We shall prove that $G=AB$. Suppose, on the contrary, that $G\neq AB$. Then, G possesses a subnormal subgroup C such that $G=ABC$, $AB\cap C=1$ and $C\neq 1$. Let $c\in C$, $c\neq 1$, and let N be a normal subgroup of C maximal with respect to the property that $c\notin N$. Since, by Lemma 1, C is an nS-group, there is a subnormal subgroup D of C such that $C=ND$ and $N\cap D=1$, and it follows that $D\simeq C/N$. Now by the choice of N, every nontrivial normal subgroup of C/N contains the element cN. Hence C/N, and therefore D, possesses a unique minimal normal subgroup. Let the minimal normal subgroup of D be M. Then $M\neq 1$ and M is subnormal in G. If M is abelian, we have $M\subseteq B$. On the other hand, if M is nonabelian, then by Lemma 2, M^G is minimal normal in G, and so $M\subseteq M^G\subseteq A$. In either case, $AB\cap C\neq 1$, a contradiction. Hence $G=AB$.

Next we observe that A is a direct product of minimal normal subgroups of G, and $A\cap B \lhd G$. It follows easily that $A=A_1\times(A\cap B)$ where A_1 is a direct product of some minimal normal subgroups M_λ $(\lambda\in\Lambda)$ of G. We write $A_1=\prod^{\times}_{\lambda\in\Lambda} M_\lambda$. Hence $G=AB=A_1B$, and since $A_1\cap B\subseteq A_1\cap (A\cap B)=1$, we have

$$G = A_1 \times B = \left(\prod_{\lambda\in\Lambda}^{\times} M_\lambda\right) \times B.$$

Since a normal subgroup of a direct factor of G is normal in G, each M_λ is simple.

Finally, the subgroup B is locally nilpotent (from its definition) and is an nS-group, by Lemma 1. The result then follows from Černikov's theorem cited above.

REFERENCES

1. S. N. Černikov, *Groups with systems of complemented subgroups*, Mat. Sb. **35** (**77**) (1954), 93–128; English transl., Amer. Math. Soc. Transl. (2) **17** (1961), 117–152. MR **16**, 565; **23** #A1714.

2. C. Christensen, *Complementation in groups*, Math. Z. **84** (1964), 52–69. MR **29** #1263.

3. ———, *Groups with complemented normal subgroups*, J. London Math. Soc. **42** (1967), 208–216. MR **34** #7648.

4. N. T. Dinerstein, *Finiteness conditions in groups with systems of complemented subgroups*, Math. Z. **106** (1968), 321–326. MR **38** #3340.

5. J. Wiegold, *On direct factors in groups*, J. London Math. Soc. **35** (1960), 310–320. MR **24** #A164.

6. D. I. Zaicev, *On normally factorizable groups*, Dokl. Akad. Nauk SSSR **197** (1971), 1007–1009 = Soviet Math. Dokl. 12 (1971), 601–604. MR **44** #2820.

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