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ON THE FUNDAMENTAL DOMAIN OF A DISCRETE GROUP

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ABSTRACT. Gel'fand, Graev, and Pjateckiĭ-Šapiro proposed a method to construct the fundamental domain of a discrete group acting on certain metric spaces. A counterexample is given to show that this construction sometimes fails.

Let W be a topological space, and let Γ be a discrete group of homeomorphisms of W. The group Γ is discrete if for every $w \in W$ the set of points γw with $\gamma \in \Gamma$ has no limit point in the space W. A *fundamental domain* of the group Γ in W is an open set $F \subset W$ such that (1) if γ_1, $\gamma_2 \in \Gamma$ and $\gamma_1 \neq \gamma_2$, then $\gamma_1 F \cap \gamma_2 \bar{F} = \varnothing$, and (2) W is the union of the sets $\gamma \bar{F}$ with $\gamma \in \Gamma$.

Suppose that W is a locally compact metric space, that the metric d on W has the "convexity" property that for every $w_0, w_1 \in W$ there exists $w_2 \in W$ such that $d(w_0, w_2) = d(w_1, w_2) = \frac{1}{2} d(w_0, w_1)$, and that $d(w_0, w_1) = d(\gamma w_0, \gamma w_1)$ for every $\gamma \in \Gamma$. Choose a point $w_0 \in W$ such that $\gamma w_0 \neq w_0$ for all $\gamma \in \Gamma$ with $\gamma \neq e$. Define

$$A = \{w \in W \mid d(w, w_0) < d(w, \gamma w_0) \text{ for all } \gamma \in \Gamma, \gamma \neq e\},$$
$$B = \{w \in W \mid d(w, w_0) \leqq d(w, \gamma w_0) \text{ for all } \gamma \in \Gamma\}.$$

In [**1**, p. 6] it is claimed that $\bar{A} = B$ and that A is a fundamental domain of Γ in W. But the following example shows that both these claims are false. (If indeed $\bar{A} = B$ and if also every closed and bounded set in W is compact, then the argument in [**1**] does prove that A is a fundamental domain of Γ in W.)

Take the plane R^2 and define the distance between two points by

$$d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|.$$

This metric is equivalent to the usual metric on the plane. Let $\Gamma = \{\gamma_n | n = 0, \pm 1, \pm 2, \cdots\}$ be the discrete group of translations $\gamma_n(x, y) = (x+n, y+n)$. Then $\gamma_n(0, 0) = (n, n) \neq (0, 0)$ for all $n \neq 0$, and

$$A = \{(x, y) \mid |x| + |y| < |x - n| + |y - n| \text{ for all } n \neq 0\}.$$

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For $n\neq 0$, let $A_n=\{(x,y)|\ |x|+|y|<|x-n|+|y-n|\}$. If $n>0$, then $A_n=\{(x,y)|x<n$ and $y<\min(n,n-x)\}$ and $A_{-n}=\{(x,y)|x>-n$ and $y>\max(-n,-n-x)\}$. Then $A_n\cap A_{-n}$ is the open hexagon

$$\{(x,y)\mid -n<x<n \text{ and } \max(-n,-n-x)<y<\min(n,n-x)\}.$$

Therefore, $A=\bigcap_{n=1}^{\infty}(A_n\cap A_{-n})$ is simply the open hexagon

$$A=\{(x,y)\mid -1<x<1 \text{ and } \max(-1,-1-x)<y<\min(1,1-x)\}.$$

Similarly,

$$B=\{(x,y)\mid |x|+|y|\leqq |x-n|+|y-n| \text{ for all } n\}$$

is the closed hexagon $\bar{A}$ together with the entire second and fourth quadrants of the plane. Clearly, $\bar{A}\neq B$ and A is not a fundamental domain of Γ in R^2. Note that the infinite strip $F=\{(x,y)|-1-x<y<1-x\}$ is a fundamental domain, and that $A\subset F\subset \bar{F}\subset B$.

Reference

1. I. M. Gel′fand, M. I. Graev and I. I. Pjateckiĭ-Šapiro, *Generalized functions. Vol. 6: Theory of representations and automorphic functions*, "Nauka", Moscow, 1966; English transl., Saunders, Philadelphia, Pa., 1969. MR **36** #3725; **38** #2093.

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