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CHEBYSHEV APPROXIMATION WITH A NULL SPACE

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ABSTRACT. Chebyshev approximation involving continuous functions vanishing on a closed set V is considered. The approximating families studied have the betweenness property. Examples are given of such families. A necessary and sufficient condition for uniqueness of best approximations is obtained.

Let X be a compact space and V be a closed subset of X. Let $C(V, X)$ be the space of continuous functions on X which vanish on V. For $g \in C(V, X)$ define

$$\|g\| = \sup\{|g(x)| : x \in X\}.$$

Let $\mathcal{G}$ be a subset of $C(V, X)$. The Chebyshev problem is: Given $f \in C(V, X)$, find G^* in $\mathcal{G}$ to minimize $e(G) = \|f - G\|$. Such an element G^* is called a best approximation in $\mathcal{G}$ to f on X.

At least two cases of such a problem arise naturally, namely approximation with functions vanishing at zero and approximation with functions decaying to zero at infinity.

A seemingly more general problem is to approximate with functions which agree with $v \in C(X)$ on a closed subset V of X. This problem, however, reduces to the previous problem if we subtract v from all functions.

We consider the best approximation problem and in particular the uniqueness problem if $\mathcal{G}$ has the betweenness property [**1**].

DEFINITION. A family $\mathcal{G}$ of continuous functions is said to have the *betweenness property* if for any two elements G_0 and G_1, there exists a λ-set $\{H_\lambda\}$ of elements of $\mathcal{G}$ such that $H_0 = G_0$, $H_1 = G_1$ and for all $x \in X$, $H_\lambda(x)$ is either a strictly monotonic continuous function of λ or a constant, $0 \leqq \lambda \leqq 1$.

EXAMPLE. Let $\mathcal{G}$ be a linear subspace of $C(V, X)$, then $\mathcal{G}$ has the betweenness property, for a λ-set is given by $H_\lambda = \lambda G_1 + (1-\lambda) G_0$.

EXAMPLE. Let P be a linear subspace of $C(V, X)$ and Q a linear subspace of $C(X)$ then $\mathcal{G} = \{p/q : p \in P, q \in Q, q > 0\}$ is in $C(V, X)$ and has the betweenness property [**1**, 152].

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LEMMA. *Let σ be a continuous strictly monotonic mapping of the real line into the real line such that $\sigma(0)=0$.*

Let $\mathscr{G} \in C(V, X)$ have the betweenness property. Then

$$\phi(\mathscr{G}) = \{\sigma(G): G \in \mathscr{G}\} \subset C(V, X)$$

and has the betweenness property.

PROOF. Let $\{H_\lambda\}$ be a λ-set for G_0 and G_1. Then $\{\sigma(H_\lambda)\}$ is a λ-set for $\sigma(G_0)$ and $\sigma(G_1)$.

LEMMA. *Let $\mathscr{G} \subset C(W, X)$ have the betweenness property and $s \in C(V, X)$. Then the set $s\mathscr{G}$ (consisting of products of s and elements of $\mathscr{G}$) is in $C(W \cup V, X)$ and has the betweenness property.*

The previous theory obtained for betweenness [**1**] gives a characterization of best approximations and an error-determining set on which best approximations agree. We must, however, develop a new theory for uniqueness.

DEFINITION. $\mathscr{G} \subset C(V, X)$ has zero-sign compatibility with null space V if for any two distinct elements G and H, any closed subset Z of the zeros of $G-H$ which contains no points of V and for any $s \in C(V, X)$ taking values -1 or $+1$ on Z, there exists $F \in \mathscr{G}$ such that

$$(*) \qquad \operatorname{sgn}(F(X) - G(x)) = s(x), \qquad x \in Z.$$

THEOREM. *Let $\mathscr{G} \subset C(V, X)$ have the betweenness property. A necessary and sufficient condition that for every $f \in C(V, X)$ a best approximation is unique is that $\mathscr{G}$ have zero-sign compatibility with null space V.*

The proof is the same as the proof of the corresponding result in [**1**].

The case where $\mathscr{G}$ is a finite-dimensional linear family is of particular interest. It can be shown using the above theorem that a necessary and sufficient condition for uniqueness is that $\mathscr{G}$ be a Haar subspace on $X \sim V$. Independent proofs of necessity and sufficiency are given in [**3**], [**2**], respectively.

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